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MIXED SIGNAL LETTER

On a multivibrator that employs a fractional capacitor

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Abstract The simple free running multivibrator built around a single fractional capacitor is examined in this letter. Equations for the oscillation frequency of the multivibrator are derived taking into account the positive feedback factor around the multivibrator. We show that the use of the fractional capacitance allows the multivibrator to have very high frequencies of oscillation for reasonable time constants used. PSPICE simulation and experimental results demonstrate the analysis with an approximation to a fractional capacitor that yields a result, which is at least 1000 times in frequency compared to if a normal capacitor of the same value was employed.

Keywords Multivibrators · Fractional Calculus · Circuit theory · Oscillators

1 Introduction

Interest in fractional order circuits which employ fractional capacitors with impedance proportional to $Z(s) = 1/Cs^{\alpha}$ $(0 < \alpha < 1)$ is not new [1]. However, there has been renewed interest in the topic particularly exploring fractional order filters and sinusoidal oscillators [2, 3] motivated by the first reported implementation of a physical fractional capacitor [4] which paves the way forward for practical applications. Most recently, fractional order behaviour has been observed in fruits and vegetables [5, 6] and its use in other areas such as electromagnetics [7] is being explored.

For decades, implementations in the analog world of fractional capacitors have been based on the approximation using passive elements obtained through partial fraction expansion [1] or self similar trees [8]. A particular advantage of fractional order circuits over normal integer order ones was first highlighted in [9]. In particular fractional order *RC* sinusoidal oscillators were shown to have an oscillation frequency proportional to $(\frac{1}{RC})^{1/\alpha}$ indicating the possibility of obtaining very high frequencies (for small α) independent of the *RC* time constant.

In this letter we examine the effect of using a fractional order capacitor in the popular single opamp multivibrator. We demonstrate that the fractional order capacitor has the ability to increase the oscillator frequency significantly all the while using reasonable circuit time constants. The results can be easily extended to other forms of multivibrator circuits. Because most multivibrators employ only one capacitor, closed form formulae are possible to derive.

2 The fractional multivibrator

The proposed circuit is shown in Fig. 1 and represents the basic free-running multivibrator without amplitude limiting control. Let $\mathbf{C}_{\alpha}$ be the fractional capacitor whose impedance $Z(s) = 1/Cs^{\alpha}$ where $0 < \alpha < 1$. Resistors R_2 and R_3 set the voltage v_z at the non-inverting node, and also determine the oscillation frequency. We assume that the

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opamp employed here has a gain bandwidth product that far exceeds that of the desired oscillation frequency. For the simple case of $\alpha = 1$ it is well known that the period of oscillation T of this multivibrator is linearly related to the time constant $\tau = RC$ by

$$\frac{T}{2} = \tau \ln\left(\frac{1+\beta}{1-\beta}\right) \qquad (1)$$

where $\beta = R_2/(R_2 + R_3)$. Now for $\alpha < 1$, an examination of the step response of $v_z(t)$ as obtained from Eq. 23 of [10] of the fractional RC circuit during half a period, shows that the period T and τ are related by the closed form expression

$$\frac{1-\beta}{1+\beta} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{\tau}\right)^n \left(\frac{T}{2}\right)^{n\alpha}}{\Gamma(n\alpha + 1)} \qquad (2)$$

where $\Gamma(\cdot)$ is the gamma function and we have made the substitution $a = -\frac{1}{\tau}$, $t = \frac{T}{2}$ and $q = \alpha$ into (10b) of [10] and used it in (23) of [10]. For the special case of $\alpha = 1/2$ (capacitor of order 1/2), (2) reduces to

$$\frac{1-\beta}{1+\beta} = e^{\frac{T}{2\tau^2}} \cdot \operatorname{erfc}\left(\frac{1}{\tau}\sqrt{\frac{T}{2}}\right) \qquad (3)$$

where erfc(·) is the compliment of the error function. A simple unit plot showing the relationship between the period T and τ at two different values of β are shown in Fig. 2. For the case of $\beta = 0.3$ the solid line obtained from Eq. 3 is compared to the dashed straight line obtained from (1). It can be seen, that for certain values of τ (in this case less than 1.38 for $\beta = 0.3$ and 0.54 for $\beta = 0.5$), the period of the fractional multivibrator is much less than that of the normal multivibrator. The difference becomes even more apparent when we use a more practical example such as $\tau = 1$ ms, then the $f_{osc}(fractional) = 1.12$ MHz compared to $f_{osc}(normal) = 807$ Hz. This represents an order in excess of 1000. The reason for this is because the step response of the fractional order RC circuit rises much faster than the conventional exponential response. Likewise for $\beta = 0.5$, and $\tau = 1$ ms, then $f_{osc}(fractional) = 244.3$ kHz compared to $f_{osc}(normal) = 455.1$ Hz or a factor still in excess of 500. It is thus clear that for low values of β the fractional multivibrator can produce much higher frequencies than its integer order counterpart.

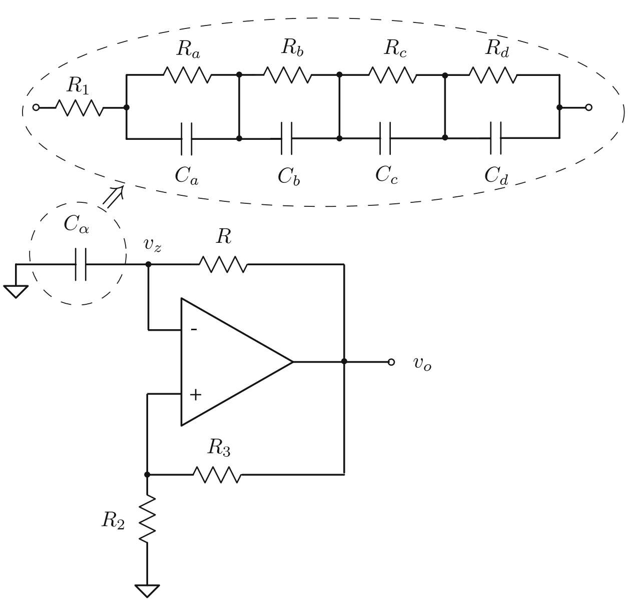


Fig. 1 The free running multivibrator with fractional capacitor $\mathbf{C}_\alpha$. In the inset a RC realization of the fractional capacitor $\mathbf{C}_\alpha$ using the method of [1]

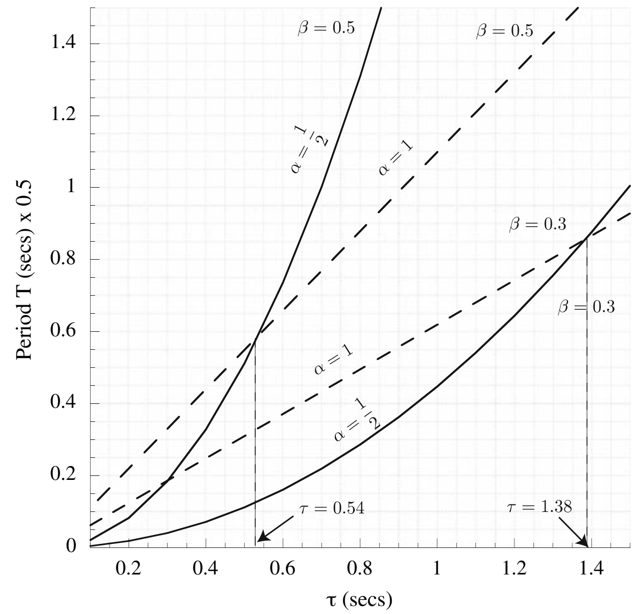
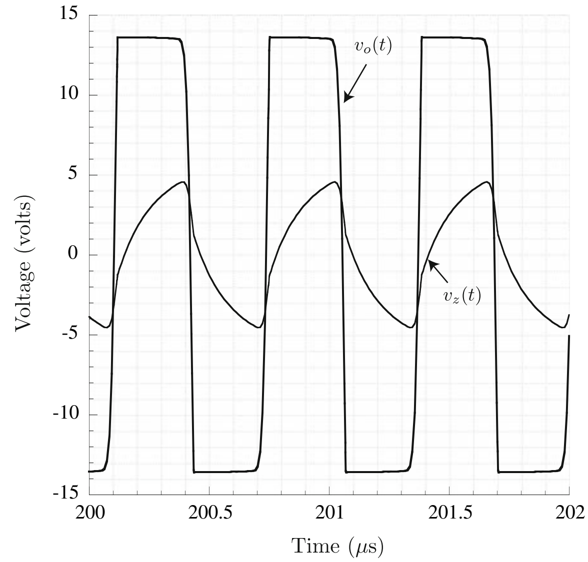


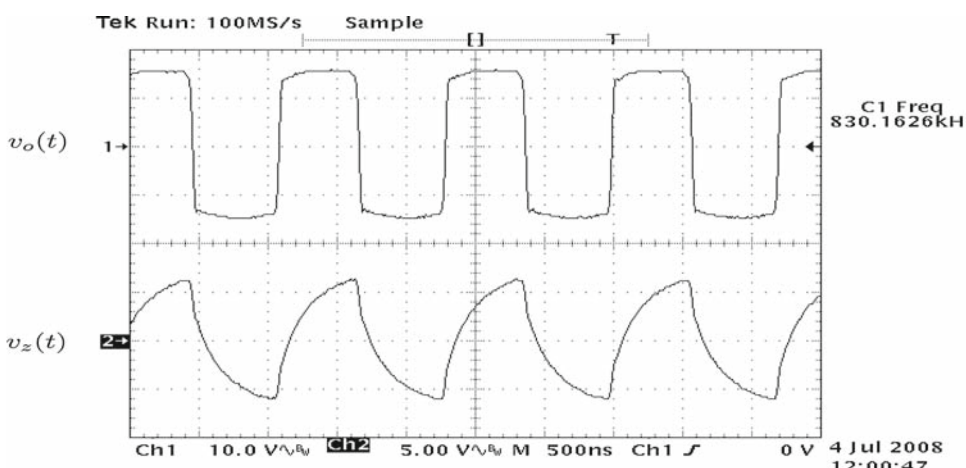
Fig. 2 Plot of period T versus τ to two values of β for the multivibrator. The dashed line represents $\alpha = 1$ or the result with a normal capacitor and the solid line represents the result due to the fractional capacitor $\alpha = 0.5$

3 Simulations and experimental results

To verify the above results for the circuit of Fig. 1, it was simulated in PSPICE and constructed on breadboard. A fractional capacitor of value 1 μF was approximated using Carlson's method [1] to a second order and realized using the partial fraction decomposition shown in the inset of Fig. 1. The opamp used was the LT1364 with a gain bandwidth product of 70 MHz and $\beta = 0.3$ was used. Resistor R was set at 1 kΩ which yielded $\tau = 1$ ms. The PSPICE results shown in Fig. 3(a) revealed the new multivibrator oscillated with a frequency of 1.57 MHz with no output amplitude stabilization. The experimental results



(a)



(b)

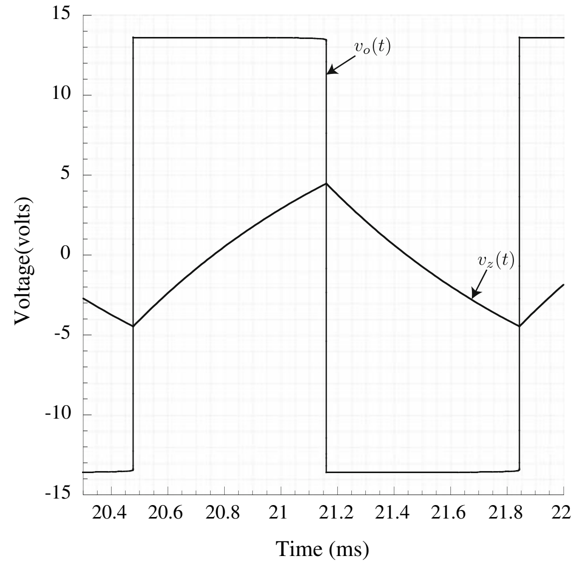
Fig. 3 **a** PSPICE Simulation of the multivibrator using an approximation to the fractional capacitor. The observed oscillation frequency was 1.57 MHz. Values used for the fractional capacitor approximation in PSPICE are $R_1 = 0.111$ kΩ, $R_a = 7.37$ kΩ, $R_b = 0.252$ kΩ, $R_c = 0.379$ kΩ, $R_d = 0.889$ kΩ and $C_a = 4.364$ nF, $C_b = 0.526$ nF, $C_c = 1.859$ nF, and $C_d = 3.375$ nF. **b** Experimental results of the multivibrator using an approximation to the fractional capacitor. The measured oscillation frequency was 830.16 kHz. Fractional capacitor values using made up 10% tolerance resistors and 20% tolerance capacitors are $R_1 = 111$ Ω, $R_a = 7.36$ kΩ, $R_b = 251$ Ω, $R_c = 378$ Ω, $R_d = 888$ Ω and $C_a = 0.0047$ μF, $C_b = 0.55$ nF, $C_c = 1.82$ nF, and $C_d = 3.3$ nF

shown in Fig. 3(b) yielded an oscillation frequency of 830.16 kHz which is less than both the PSPICE and the theoretical result. This however, is due to the fact that exact values to the approximation could not be obtained using discrete components and the low tolerance of the discrete components available. If a normal 1 μF capacitor was used the oscillation frequency obtained in PSPICE was 731.5 Hz and shown in Fig. 4(a) and in the experimental result shown in Fig. 4(b) it was 819.68 Hz. This shows the usefulness of employing the fractional capacitor, so long as the opamp is capable of handling the increased frequencies. Both values of the expected oscillation frequencies are therefore in close agreement with the predicted results within limits and by a factor of at least 1000. Note that to generate the higher oscillation frequency using a normal integer order capacitor a time constant $\tau = 0.514$ μs (rather than $\tau = 1$ ms) is needed. Finally, the results in Figs. 3 and 4 for $v_z(t)$ also display the characteristic $1 - e^t \mathrm{erfc}(\sqrt{t})$ response which can be derived from (3).

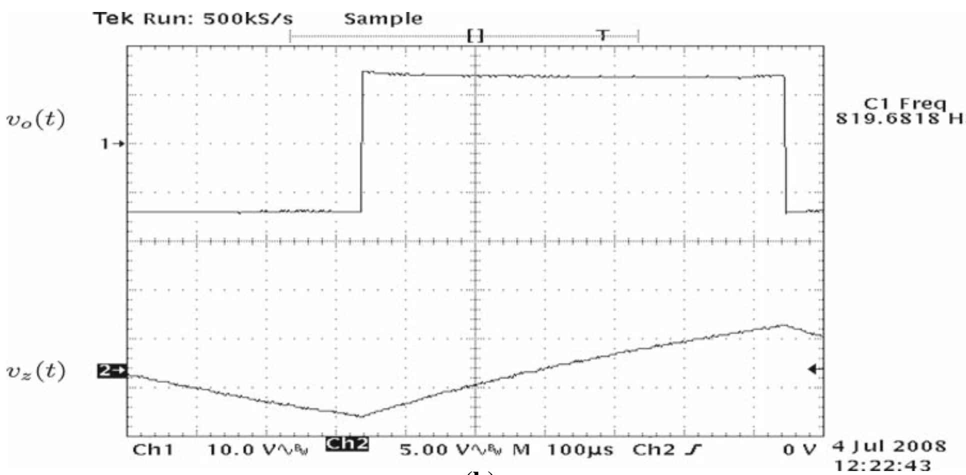
4 Discussion

The usefulness of the fractional order capacitor with multivibrators is demonstrated in this letter. The frequency of oscillation is shown to be much higher for the same time constants in the simple multivibrator. This opens up the possibility of greater frequency control in multivibrators as physical fractional capacitors soon become available.

Fig. 4 **a** PSPICE Simulation of the multivibrator using a normal 1 μF capacitor, $R = 1$ kΩ and $\beta = 0.3$. The observed oscillation frequency was 731.5 Hz. **b** Experimental results of the multivibrator using a normal 1 μF capacitor, $R = 1$ kΩ and $\beta = 0.3$. The measured oscillation frequency was 819.68 Hz



(a)



(b)

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Brent Maundy received the B.Sc. degree in Electrical Engineering and a M.Sc. degree in Electronics and Instrumentation in 1983 and 1986, respectively from the University of the West Indies, Trinidad. In 1992 he received the Ph.D. in Electrical from Dalhousie University, Halifax, Nova Scotia. He completed a one-year postdoctoral fellow at Dalhousie University where he was actively involved in its analog microelectronics group. Subsequent to that he taught at the University of the West Indies, and was a visiting Professor at the University of Louisville for 7 months. He later worked in the defense industry for 2 years on mixed signal projects. In 1997 Dr. Maundy joined the Department of Electrical and Computer Engineering at the University of Calgary where he is currently a Professor. Dr. Maundy's current research is in the design of linear circuit elements, high-speed amplifier design, active analog filters, and CMOS circuits for signal processing and communication applications. Dr. Maundy is a member of the IEEE and is a past Associate Editor for the IEEE Transactions on Circuits and Systems I.

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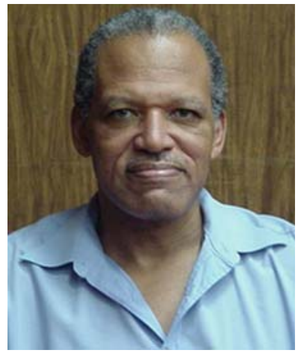

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