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LETTER TO THE EDITOR

An expression for the voltage response of a current-excited fractance device based on fractional-order trigonometric identities

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SUMMARY

We report a closed-form expression of the voltage response of a current-excited fractance device. The derived simple expression is made possible following the introduction of the generalized sine and cosine functions ($\sin_\alpha(t)$ and $\cos_\alpha(t)$), which are valid on any fractional-order surface and tend to the normal $\sin(t)$ and $\cos(t)$ at $\alpha=1$ or asymptotically as $t\to\infty$. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Concepts of Fractional Calculus have been developed by mathematicians quite a long time ago [1]. In particular, the Riemann–Liouville definition of the fractional integral and fractional derivative of a function $f(t)$ are given by

$$J^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \quad \alpha\in\mathbb{R}^+, \quad t>0 \tag{1a}$$

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t)=D^\alpha f(t)=D^m(J^{m-\alpha} f(t)), \quad m-1<\alpha\leqslant m \tag{1b}$$

and the Laplace transform of the fractional derivative (left-inverse operator) is given by

$$L(D^\alpha f(t))=s^\alpha F(s)-\sum_{k=0}^{m-1} D^k J^{(m-\alpha)} f(0^+)s^{m-1-k}, \quad m-1<\alpha\leqslant m \tag{2}$$

More details on fractional calculus can be found in [2, 3].

On the other hand, an electrical impedance $Z(s)$ is defined in the complex frequency domain ($s=\mathrm{j}\omega$) as the ratio of the measured voltage to the applied current; i.e. $Z(s)=V(s)/I(s)$. A '*Fractance device*' is one whose impedance is proportional to $s^{\pm\alpha}$; $0<\alpha<1$ [4–6]. The three special cases where $\alpha=0$, 1 or -1 correspond respectively to the three well-known elements the resistor, the inductor and the capacitor although it was noted in [7] that accurate modelling of

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dielectric-based capacitors implies that their impedance should be given as $Z_c = 1/Cs^\alpha$ and $\alpha < 1$. Note that $(j\omega)^\alpha = \omega^\alpha e^{j\alpha\pi/2} = \omega^\alpha[\cos(\alpha\pi/2) + j\sin(\alpha\pi/2)]$ and hence the traditional 2D s-plane is more generally a 3D cone of face angle $\alpha\pi/2$ [8].

More importantly, impedance spectroscopy is a widely used technique in biochemistry and biology as a means of characterizing the properties of plant and animal tissues [9–13]. This technique is centred around exciting the tissue under consideration via an alternating current source of suitable amplitude, while measuring the resulting developed voltage over a specified frequency range of interest. Measurements are taken at steady state and the impedances of nearly all natural tissues are found to be proportional to $1/s^\alpha$; where α varies from one tissue type to the other [12]. This variation in α can also be used as a microbiological growth sensor [14]. From the measured data, an impedance model for a tissue can be constructed that always contains one or more fractance devices. Using these models, the behavior of a tissue impedance under current stimulus can be found using time-domain numerical simulations techniques [15].

In this letter, we propose a closed-form expression for the voltage $V(t)$ developed across a fractance device excited by a current $I(t)$. This expression accounts for the transient behavior and the steady-state behavior can be found taking the limit of $V(t)$ as $t \to \infty$. The key to this closed-form expression is the definition of the generalized trigonometric functions of order α; $\cos_\alpha(t)$ and $\sin_\alpha(t)$ both of which tend to the normal trigonometric functions as $t \to \infty$ or at $\alpha = 1$. The response of a fractance device to a triangle-wave current excitation in a closed-form is also given.

2. TRIGONOMETRIC FUNCTIONS OF ORDER α

The generalized exponential function of order $\alpha(E_\alpha^t)$ is defined as [1]

$$E_\alpha^t = \sum_{k=0}^{\infty} e_{k-\alpha}^t = \sum_{k=0}^{\infty} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} \tag{3}$$

where E_α^t is also equal to the fractional derivative of order α of the exponential function e^t; i.e. $E_\alpha^t = (d^\alpha/dt^\alpha)(e^t)$. On the other hand using Euler's identity we can write

$$E_\alpha^{jt} = \cos_\alpha(t) + j\sin_\alpha(t) \tag{4}$$

where $\cos_\alpha(t)$ and $\sin_\alpha(t)$ are then the generalized cosine and sine functions of order α. This allows us now to define these two functions as

$$\sin_\alpha(t) = \frac{1}{2j}(E_\alpha^{jt} - E_\alpha^{-jt}) = \sum_{k=0}^{\infty} e_{k-\alpha}^t \sin(k-\alpha)\frac{\pi}{2} \tag{5a}$$

$$\cos_\alpha(t) = \frac{1}{2}(E_\alpha^{jt} + E_\alpha^{-jt}) = \sum_{k=0}^{\infty} e_{k-\alpha}^t \cos(k-\alpha)\frac{\pi}{2} \tag{5b}$$

Note that setting $\alpha = 1$ in (5a) for example leads to

$$\sin_1(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sin\left(k\frac{\pi}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} \sin\left(k\frac{\pi}{2}\right) = \sin(t)$$

as expected. Therefore, the normal $\sin(t)$ and $\cos(t)$ functions are obtained from their generalized fractional-order ones by setting $\alpha = 1$. Figure 1(a) shows a 3D plot of the $\cos_\alpha(t)$ versus t and α. The difference between the fractional and the normal cosine functions; i.e. $\cos_\alpha(t) - \cos_1(t)$, for different values of α, is plotted in Figure 1(b) from which it is also clear that as $t \to \infty$, $\cos_\alpha(t)$ asymptotically tends to $\cos(t)$. The same applies for $\sin_\alpha(t)$.

It is also important to highlight the following:

(1) The sum $S_1 = \sin_\alpha^2(t) + \cos_\alpha^2(t) \neq 1$. However, S_1 is always real and $\lim_{t\to\infty} S_1 = 1$ as expected. Figure 2 is a plot of S_1. Note that $S_1 = 1$ only at $\alpha = 1$ that corresponds to the classical trigonometric identity $\sin^2(t) + \cos^2(t) = 1$.

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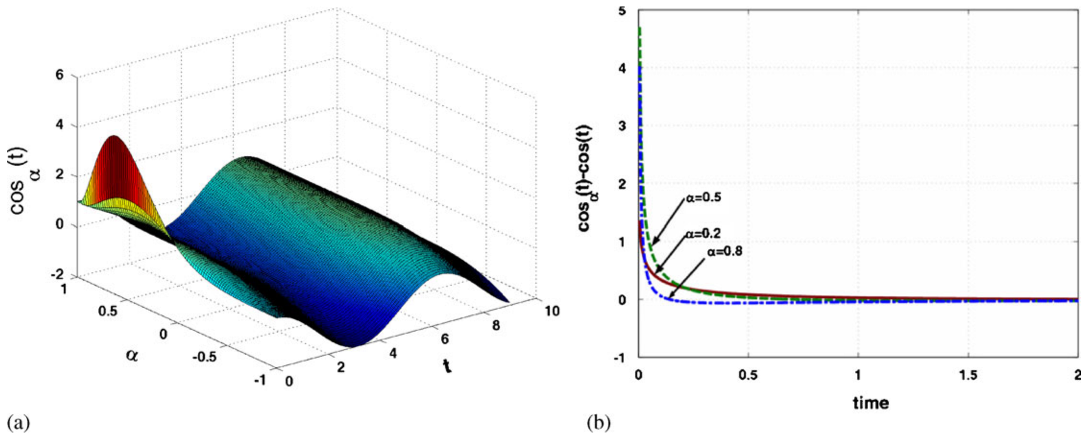

Figure 1. (a) 3D plot of the fractional cosine function $\cos_\alpha(t)$ and (b) plot of the difference $\cos_\alpha(t)-\cos(t)$.

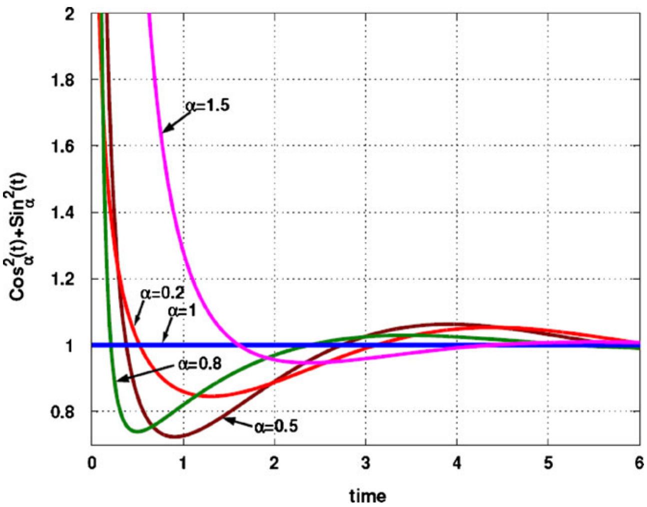

Figure 2. Plot of $\sin^2_\alpha(t)+\cos^2_\alpha(t)$ for different α's.

(2) The function $F_1=\sin_\alpha(2t)\neq 2\sin_\alpha(t)\cos_\alpha(t)$. Also $\cos_\alpha(2t)\neq\cos^2_\alpha(t)-\sin^2_\alpha(t)$. Figures 3(a) and (b) clearly show that only at $t\to\infty$ does the asymptotic value of $\sin_\alpha(2t)$ tend to $2\sin(t)\cos(t)$ and of $\cos_\alpha(2t)$ tend to $\cos^2(t)-\sin^2(t)$.
(3) The function $F_2=\sin_\alpha(t)\cos_\alpha(t)$ is equal to $C\cdot t^{k-2\alpha}$. The value of C is as given in Table I.

3. VOLTAGE RESPONSE OF A CURRENT-EXCITED FRACTANCE DEVICE

A fractance device (also known as the 'constant phase element (CPE)' [12, 14]) is an element whose impedance is $Z(\mathrm{j}\omega)=ks^\alpha$, where α is of arbitrary positive or negative value.‡ Assuming a sinusoidal current of the form $I(t)=I_o\sin(\omega t)$ is used to excite this impedance, then the voltage developed across this device would be given by

$$V(t)=L^{-1}[ks^\alpha I(s)]=kD^\alpha(I(t))=kI_o\omega^\alpha\left[\sin_\alpha(\omega t)\cos\left(\frac{\alpha\pi}{2}\right)+\cos_\alpha(\omega t)\sin\left(\frac{\alpha\pi}{2}\right)\right] \quad (6)$$

where $\sin_\alpha(\omega t)$ and $\cos_\alpha(\omega t)$ are as defined respectively in Equation (5).

‡When α is restricted to being negative, the fractance device is also known as the 'fractional capacitor' [16].

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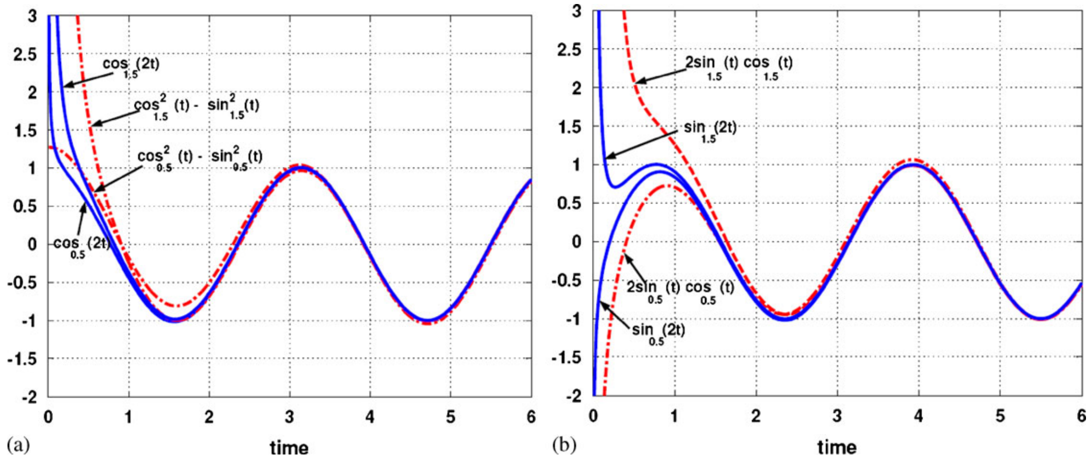

Figure 3. (a) Plot of $\cos_\alpha(2t)$ for different α's and (b) plot of $\sin_\alpha(2t)$.

Table I. Values of C in $F_2=\sin_\alpha(t)\cos_\alpha(t)=C\cdot t^{k-2\alpha}$.

if $k=2n$	$C=(-1)^{n+1}(\sin\alpha\pi)\left[\sum_{i=0}^{n-1}\frac{1}{\Gamma(1+i-\alpha)\Gamma(2n-\alpha+1-i)}+\frac{1}{2(\Gamma(n-\alpha+1))^2}\right]$
if $k=2n+1$	$C=(-1)^n(\cos\alpha\pi)\sum_{i=0}^{n}\frac{1}{\Gamma(1+i-\alpha)\Gamma(2n-\alpha+2-i)}$
if $k=2n$ and $\alpha=0,1,2,\ldots$	$C=0$
if $k=2n+1$ and $\alpha=0,1,2,\ldots$	$C=(-1)^n\frac{2^{2n}}{(2n+1)!}\left(\text{note that}\sum_{i=0}^{n}\frac{1}{\Gamma(1+i)\Gamma(2n+2-i)}=\frac{2^{2n}}{(2n+1)!}\right)$
if $k=2n$ and $\alpha=0.5,1.5,2.5,\ldots$	$C=0$
if $k=2n+1$ and $\alpha=0.5,1.5,2.5,\ldots$	$C=(-1)^n\sum_{i=0}^{n}\frac{1}{\Gamma(0.5+i)\Gamma(2n+1.5-i)}+\frac{1}{2(\Gamma(n-1.5))^2}$

Figure 4 is a 3D plot of $V(t)$ in the $V-\alpha-t$ space assuming $\omega=1$ rad/s, $k=1$ and $I_o=1A$. The steady-state voltage across the fractance device is given by

$$V_{ss}(t)=\lim_{t\to\infty}V(t)=kI_o\omega^\alpha\sin\left(\omega t+\frac{\alpha\pi}{2}\right) \tag{7}$$

which clearly shows a phase angle $\theta=\alpha\pi/2$ between the exciting current and the measured voltage. It is this angle that is of particular importance in impedance spectroscopy analysis since by measuring θ, α is directly obtained [9–14].

We may also obtain a closed-form expression for $V(t)$ given other forms of exciting currents. For example, if $I(t)$ is a periodic triangle waveform with fundamental frequency $\omega_o=2\pi/T$ (which can be represented as a sum of harmonics), it can be shown that $V(t)$ in this case is given by[§]

$$\begin{aligned}V(t)&=\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^{2-\alpha}}\left[\sin_\alpha((2n-1)t)\cos\left(\frac{\alpha\pi}{2}\right)+\cos_\alpha((2n-1)t)\sin\left(\frac{\alpha\pi}{2}\right)\right]\\&\cong\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^{2-\alpha}}\sin\left((2n-1)t+\frac{\alpha\pi}{2}\right)\end{aligned} \tag{8}$$

where $\sin_\alpha(\omega t)$ and $\cos_\alpha(\omega t)$ are as defined respectively in Equation (5). Figure 5 shows this waveform of $V(t)$ for different values of α. Note that at $\alpha=0$, $V(t)$ is also a triangle waveform following $I(t)$ while at $\alpha=1$, $V(t)$ is a square-wave indicating that $I(t)$ is order-one differentiated. The $V(t)$ waveform at half differentiation ($\alpha=0.5$) is seen to be somewhere in between.

[§]Assuming for simplicity $k=1$.

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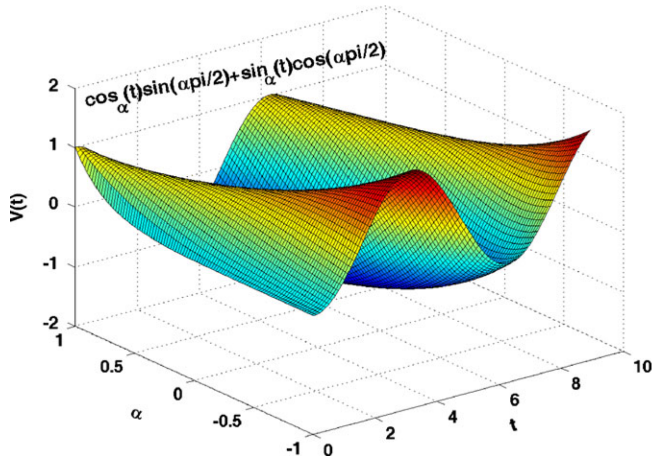

Figure 4. 3D plot of the voltage $V(t)$ across a fractance device excited by a sinusoidal current.

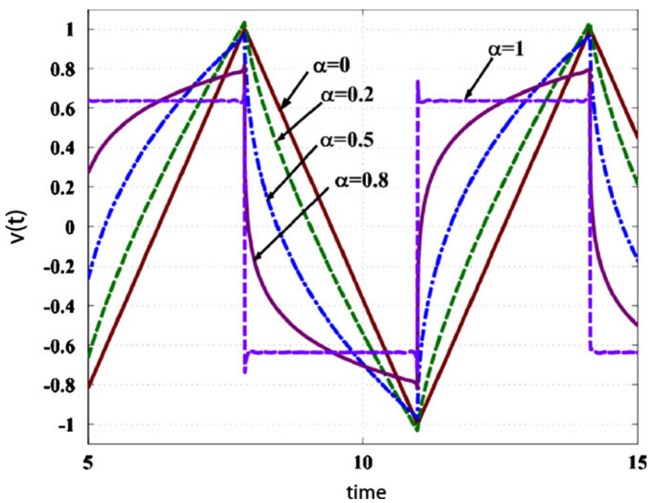

Figure 5. $V(t)$ across a fractance device for different α's if $I(t)$ is a triangle waveform obtained from numerical integration of (8).

4. CONCLUSION

In this letter, we have shown that the expression for the voltage developed across a fractance device due to a current excitation can be obtained in a closed form by explicitly using the defined fractional trigonometric functions of order α. Apart from the crucial application in bio-impedance spectroscopy, it is important to note that circuits and systems that employ fractance devices (or the more restricted fractional capacitors) are attracting attention due to their unique features. Among these features are very accurate feedback controllers [17] as well as oscillators with very high oscillation frequencies that are independent of any RC time constant [18, 19].

REFERENCES

1. Oldham KB, Spanier J. *Fractional Calculus*. Academic Press: New York, 1974.
2. Samko SG, Kilbas AA, Marichev OI. *Fractional Integrals and Derivatives*: *Theory and Application*. Gordon & Breach: Amsterdam, 1987.
3. Miller KS, Ross B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley: New York, 1993.

DOI: 10.1002/cta

4. Nakagawa M, Sorimachi K. Basic characteristics of a fractance device. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences* 1992; **E75**:1814–1819.
5. Sugi M, Hirano Y, Miura YF, Saito K. Simulation of fractal immittance by analog circuits: an approach to the optimized circuits. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences* 1999; **E82**:1627–1634.
6. Krishna BT, Reddy K. Active and passive realization of fractance device of order 1/2. *Active and Passive Electronic Components* 2008; DOI: 10.1155/2008/369421.
7. Westerlund S. Capacitor theory. *IEEE Transactions on Dielectrics and Electrical Insulation* 1994; **1**:826–839.
8. Radwan AG, Soliman AM, Elwakil AS. On the stability of fractional-order systems. *Chaos Solitons and Fractals* 2009; **40**:2317–2328.
9. Elwakil AS, Maundy B. Extracting the cole–cole impedance model parameters without direct impedance measurement. *Electronics Letters* 2010; **46**:1367–1368.
10. Barsoukov E, Macdonald J. *Impedance Spectroscopy: Theory, Experiment, and Applications*. Wiley: New York, 2005.
11. Inaba A, Manabe T, Tsuji H, Iwamoto T. Electrical impedance analysis of tissue properties associated with Ethylene induction by electric circuits in Cucumber. *Plant Physiology* 1995; **107**:195–205.
12. Jesus IS, Machado JA, Cunha J. Fractional electrical impedances in botanical elements. *Journal of Vibration and Control* 2008; **14**:1389–1402.
13. Tang C, You F, Cheng G, Gao D, Fu F, Dong X. Modeling the frequency dependence of the electrical properties of the live human skull. *Physiological Measurement* 2009; **30**:1293–1301.
14. Biswas K, Sen S, Dutta P. A constant phase element sensor for monitoring microbial growth. *Sensors and Actuators B* 2006; **119**:186–191.
15. Wu J, Chen C. A new operational approach for solving fractional calculus and fractional differential equations numerically. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences* 2004; **E87**:1077–1082.
16. Ahmed W, Elkhazali R, Elwakil AS. Fractional-order Wien-bridge oscillator. *Electronics Letters* 2001; **37**:1110–1112.
17. Ortigueira MD. An introduction to the fractional continuous-time linear systems: the 21st century systems. *IEEE Circuits and Systems Magazine* 2008; **8**:19–26.
18. Radwan AG, Soliman AM, Elwakil AS. Fractional-order sinusoidal oscillators: Four practical circuit design examples. *International Journal of Circuit Theory and Applications* 2008; **36**:473–492.
19. Maundy B, Elwakil AS, Gift S. On a multivibrator that employs a fractional capacitor. *Journal of Analog Integrated Circuit and Signal Procesing* 2010; **62**:99–103.

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