The Generalized Exponential Function and Fractional Trigonometric Identities

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Abstract— **In this work, we recall the generalized exponential function in the fractional-order domain which enables defining generalized cosine and sine functions. We then re-visit some important trigonometric identities and generalize them from the narrow integer-order subset to the more general fractional-order domain. Generalized hyperbolic function relations are also given.**

I. Introduction

Concepts of Fractional Calculus have been developed by mathematicians quite a long time ago [1]. In particular, the Riemann-Liouville definition of the fractional integral and fractional derivative of order α for a function $f(t)$ are given respectively by

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau,\ \alpha\in\mathbb{R}^+ \quad (1a)$$

$$D^{\alpha}f(t)=D^m\left(J^{m-\alpha}f(t)\right),\ m-1<\alpha\leq m \quad (1b)$$

and the Laplace transform of the fractional derivative (left-inverse operator) is given by

$$L\left(D^{\alpha}f(t)\right)=s^{\alpha}F(s)-\sum_{k=0}^{m-1}D^kJ^{(m-\alpha)}f(0^+)s^{m-1-k} \quad (2)$$

Nevertheless, improvements and new relationships continue to be introduced in the literature [2], [3]. It is unfortunate that these concepts have not found their way to non-mathematicians except relatively recently. In particular, control system theorists were notably the first to adopt and convey fractional operators [4] and continue to do so [5]-[9]. On the electrical circuit design front, the awareness of the fractional-order domain is in its infancy with some important work done and much more still needed [10]. In [11], the design of a non-integer order differentiator was considered while in [12] a classical Wien-bridge sinusoidal oscillator was investigated assuming its two capacitive energy storage devices were characterized by fractional impedances. A series of recent articles in [13], [14] and [15] studied in detail practical fractional-order oscillators and filters. Stability has also been studied in [16].

Despite the great effort has recently been done to generalize the circuit design theory of filters and oscillator to the fractional-order domain, it seems that interest in fractional impedance spectroscopy is more important in developing sensors and measurement techniques for biological and biochemical electronic interface systems [18]-[20]. The main potential driving force for fractional-order systems in the future seems to be in biomedical signal processing[21]. Hence, in this work we define the generalized exponential function and generalized trigonometric identities in the fractional-order domain, laying the theoretical foundation for many future applications.

II. Generalized Exponential and Trigonometric Identities

The generalized exponential function of order α (E^t_{α}) can be defined as

$$D^{\alpha}(e^t)=E^t_{\alpha}=\sum_{k=0}^{\infty}e^t_{k-\alpha}=\sum_{k=0}^{\infty}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} \quad (3)$$

and hence the generalized Euler identity becomes

$$E^{jt}_{\alpha}=\cos_{\alpha}(t)+j\sin_{\alpha}(t) \quad (4)$$

where $\cos_{\alpha}(t)$ and $\sin_{\alpha}(t)$ are then the generalized cosine and sine functions of order α which are now given respectively by

$$\sin_{\alpha}(t)=\frac{1}{2j}(E^{jt}_{\alpha}-E^{-jt}_{\alpha})=\sum_{k=0}^{\infty}e^t_{k-\alpha}\sin\left(k-\alpha\right)\frac{\pi}{2} \quad (5)$$

$$\cos_{\alpha}(t)=\frac{1}{2}(E^{jt}_{\alpha}+E^{-jt}_{\alpha})=\sum_{k=0}^{\infty}e^t_{k-\alpha}\cos\left(k-\alpha\right)\frac{\pi}{2} \quad (6)$$

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Figure 1 is a 3D plot of E^t_α versus t and α. Note in particular that:

a) at $\alpha = -1$; $E^t_{-1} = e^t - 1$
b) at $\alpha = 0$; $E^t_0 = e^t$
c) at $\alpha = 1$; $E^t_1 = e^t$
d) $E^{jt}_0 = e^{jt}$ and hence from the above expression of $\sin_\alpha(t)$ we obtain that

$$\sin_0(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\sin(k\frac{\pi}{2})=\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{(2k+1)!}\sin(k\frac{\pi}{2})=\sin(t) \tag{7}$$

as expected.

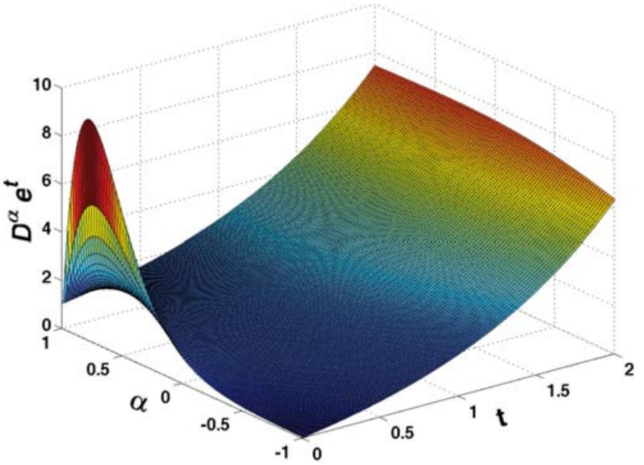


Figure 1: 3D plot of the generalized exponential function.

It is also clear that $\lim_{t\Longrightarrow\infty} E^t_\alpha = e^t$ indicating that the generalized exponential function becomes independent of α at $t \to \infty$.

Figure 2(a) is a 3D plot of the generalized cosine function ($\cos_\alpha(t)$) versus t and α. Also note that $\lim_{t\Longrightarrow\infty}\cos_\alpha(t) = \cos(t)$ and that $\cos_{-1}(t) = \cos(t) - 1$, $\cos_0(t) = \cos(t)$ and $\cos_1(t) = \cos(t)$. For more clarity, the difference between the generalized and the normal (integer-order subset) cosine functions, for different values of α, is plotted in Fig. 2(b). The difference is significant for small t and smaller α.

It is also now important to highlight the following:

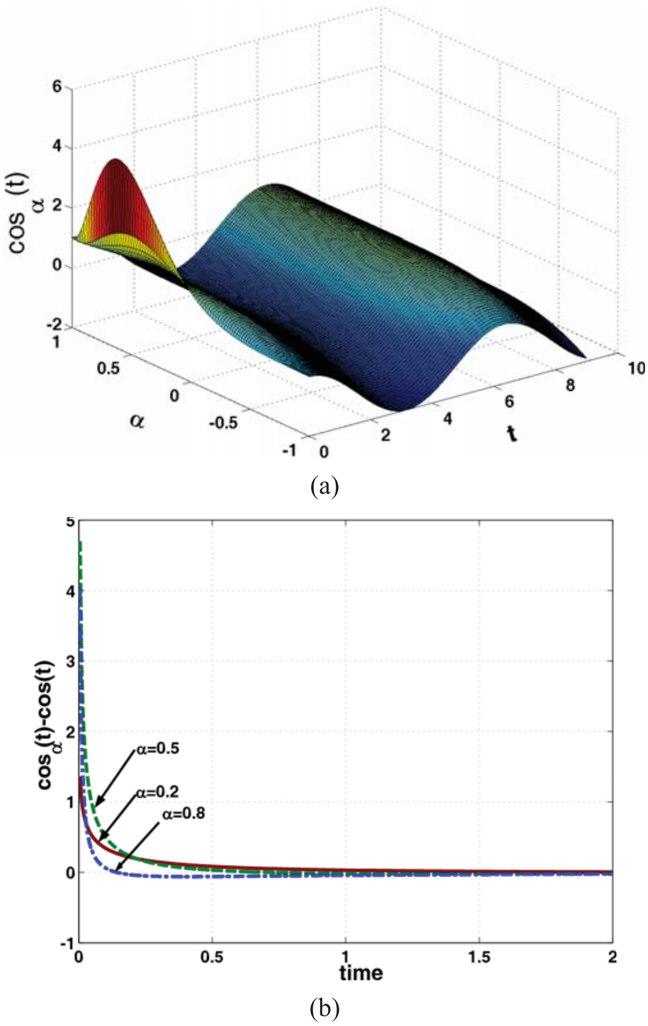
1. The product $F_1 = E^t_\alpha \cdot E^{-t}_\alpha$ does not always equal 1. F_1 is pure real for $\alpha = 0, 1, 2, ...$ and pure imaginary for $\alpha = 0.5, 1.5, ...$ For any other α, the result is complex. Figure 3(a) is a plot of $\text{Re}(F_1)$ for different values of α. It is clear that $\text{Re}(F_1) = F_1 = 1$ only for $\alpha = 1$.
2. The famous trigonometric identity $F_2 = \sin^2_\alpha(t) + \cos^2_\alpha(t)$ is not always equal to 1. However, F_2 is always real and $\lim_{t\to\infty} F_2 = 1$. Figure 3(b) is a plot of F_2. Note that $F_2 = 1$ only at $\alpha = 1$.
3. The function $F_3 = \sin_\alpha(2t) = \sum_{k=0}^{\infty} e^{2t}_{k-\alpha}\sin(k-\alpha)\frac{\pi}{2}$ is not equal to $2\sin_\alpha(t)\cos_\alpha(t)$. Also $\cos_\alpha(2t) \neq \cos^2_\alpha(t) - \sin^2_\alpha(t)$. Figures 4(a) and 4(b) clearly show that only at $t \to \infty$ does the assymptotic value of $\sin_\alpha(2t)$ tend to $2\sin_\alpha(t)\cos_\alpha(t)$ and of $\cos_\alpha(2t)$ tend to $\cos^2_\alpha(t) - \sin^2_\alpha(t)$.
4. The function $F_4 = \sin_\alpha(t)\cos_\alpha(t)$ is equal to $c \cdot t^{k-2\alpha}$. The value of c is given by

$$c = (-1)^{n+1}(\sin\alpha\pi)\cdot \sum_{i=0}^{n-1}\left[\frac{1}{\Gamma(1+i\text{-}\alpha)\Gamma(2n+1\text{-}\alpha\text{-}i)}+\frac{1}{2\Gamma(n\text{-}\alpha+1)^2}\right] \tag{8}$$

if $k = 2n$ and by

$$c = (\text{-}1)^n(\cos\alpha\pi)\sum_{i=0}^{n}\frac{1}{\Gamma(1+i\text{-}\alpha)\Gamma(2n\text{-}\alpha+2\text{-}i)} \tag{9}$$

if $k = 2n+1$.
5. $\frac{d}{dt}\cos_\alpha(t) = -\sin_\alpha(t) + \frac{\cos(\alpha\pi/2)}{t^{\alpha+1}\Gamma(-\alpha)}$.



(a)

(b)

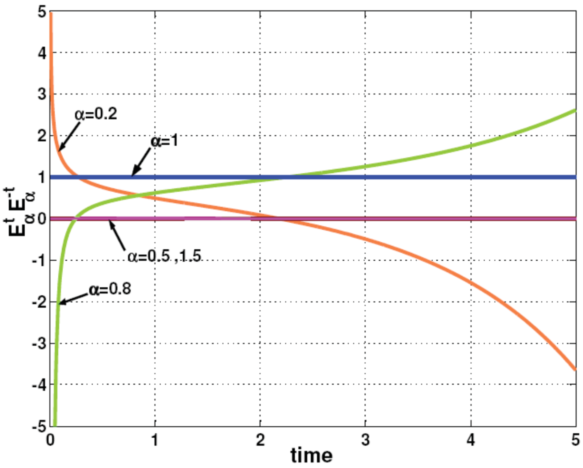
Figure 2: (a) 3D plot of the generalized cosine function $\cos_\alpha(t)$ and (b) plot of the difference $\cos_\alpha(t) - \cos(t)$.

It is also possible to generalize the hyperbolic trigonometric functions as

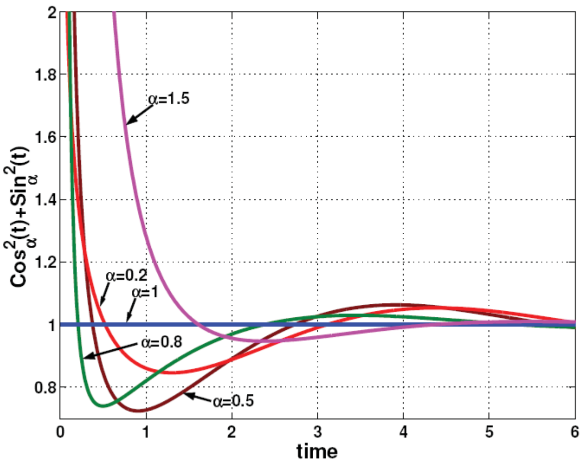
$$\cosh_\alpha(t) = \frac{1}{2}\left(E_\alpha^t + E_\alpha^{-t}\right) = \sum_{k=0}^{\infty} e_{k-\alpha}^t e^{j\frac{\pi}{2}(k-\alpha)} \cos\frac{(k-\alpha)\pi}{2} \tag{10}$$

and

$$\sinh_\alpha(t) = \frac{1}{2}\left(E_\alpha^t - E_\alpha^{-t}\right) = -j\sum_{k=0}^{\infty} e_{k-\alpha}^t e^{j\frac{\pi}{2}(k-\alpha)} \sin\frac{(k-\alpha)\pi}{2} \tag{11}$$



(a)



(b)

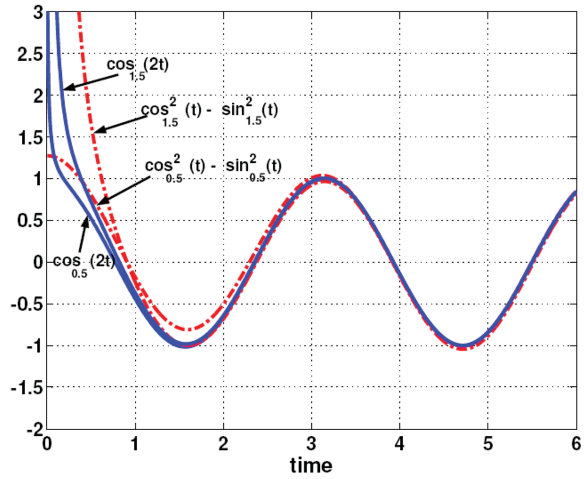
Figure 3: (a) Plot of $\mathrm{Re}(E_\alpha^t \cdot E_\alpha^{-t})$ and (b) plot of $\sin_\alpha^2(t) + \cos_\alpha^2(t)$ for different α.

Note that $\cosh_\alpha(t)$ is an even function while $\sinh_\alpha(t)$ is odd. So, for any value of α, the even power coefficients equal zero in the expansion of $\sinh_\alpha(t)$ while the odd power coefficients equal zero in the expansion of $\cosh_\alpha(t)$. Figure 5(a) shows for different values of α the Real and Imaginary parts of $\sinh_\alpha(t)$ and $\cosh_\alpha(t)$ given respectively by

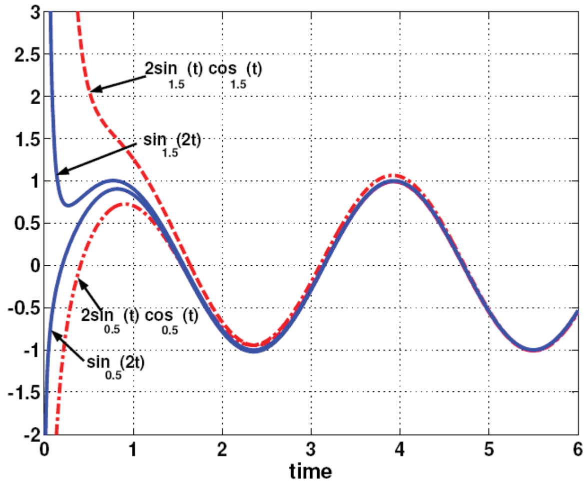
$$\mathrm{Re}(\cosh_\alpha(t)) = \sum_{k=0}^{\infty} e_{k-\alpha}^t \cos^2(k-\alpha)\frac{\pi}{2}$$
$$\mathrm{Re}(\sinh_\alpha(t)) = \sum_{k=0}^{\infty} e_{k-\alpha}^t \sin^2(k-\alpha)\frac{\pi}{2} \tag{12}$$

and

$$|\mathrm{Im}(\sinh_\alpha(t))| = |\frac{\sin(\alpha\pi)}{2}\sum_{k=0}^{\infty}(\text{-}1)^k e_{k-\alpha}^t| \tag{13}$$



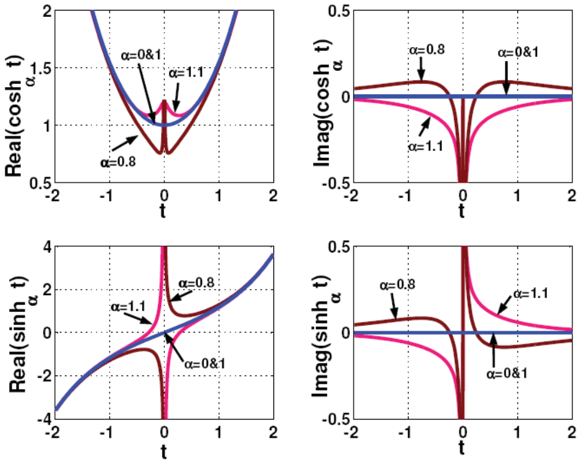
(a)



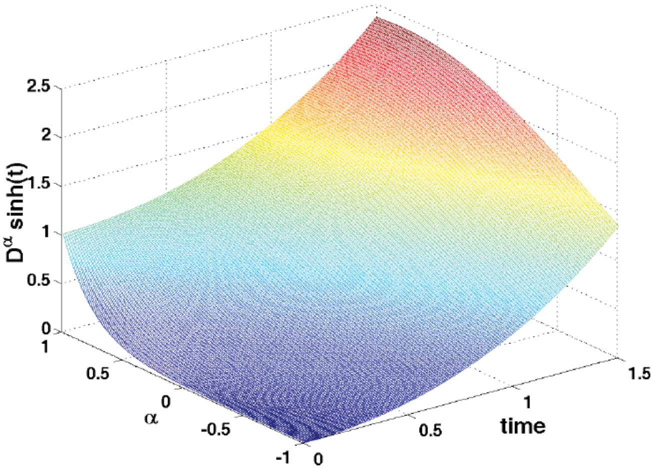
(b)

Figure 4: Simulation results of famous trigonometric identities.

Note that $\mathrm{Im}(\sinh_\alpha(t))$=$\mathrm{Im}(\cosh_\alpha(t))$ (which equals zero for any integer value of α) and that $D^\alpha(\sinh(t)) = \sum_{n=0}^{\infty}\frac{t^{2n+1-\alpha}}{\Gamma(2n+2-\alpha)}$. Figure 5(b) shows $D^\alpha(\sinh(t))$ for $\alpha \in [-1, 1]$.



(a)



(b)

Figure 5: (a) Real and Imaginary parts for $\sinh_{\alpha}(t)$ and $\cosh_{\alpha}(t)$; (b) plot of $D^{\alpha}(\sinh(t))$.

III. Conclusion

In this work we introduced important generalized trigonometric and hyper trigonometric identities. It is a general observation that fractional-order identities asymptotically tend to integer-order ones as $t \to \infty$.

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