

FIRST-ORDER FILTERS GENERALIZED TO THE FRACTIONAL DOMAIN

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Traditional continuous-time filters are of integer order. However, using fractional calculus, filters may also be represented by the more general fractional-order differential equations in which case integer-order filters are only a tight subset of fractional-order filters. In this work, we show that low-pass, high-pass, band-pass, and all-pass filters can be realized with circuits incorporating a single fractance device. We derive expressions for the pole frequencies, the quality factor, the right-phase frequencies, and the half-power frequencies. Examples of fractional passive filters supported by numerical and PSpice simulations are given.

Keywords: Analog filters; fractional calculus; fractional-order circuits; filter design.

1. Introduction

Filter design is one of the very few areas of electrical engineering for which a complete design theory exists. Whether passive or active, filters necessarily incorporate inductors and capacitors, the total number of which dictates the filter order. However, an inductor or capacitor is not but a special case of the more general so-called fractance device; which is an electrical element whose impedance in the complex frequency domain is given by $Z(s) = as^\alpha \Rightarrow Z(j\omega) = a\omega^\alpha e^{j(\pi\alpha/2)}$.^{1–5} For the special case of $\alpha = 1$ this element represents an inductor while for $\alpha = -1$ it represents a capacitor. In the range $0 < \alpha < 2$, this element may generally be considered to represent a fractional-order inductor while for the range $-2 < \alpha < 0$, it may be considered to represent a fractional-order capacitor. At $\alpha = -2$, it represents the well-known frequency-dependent negative resistor (FDNR). Although a physical

fractance device does not yet exist in the form of a single commercial device, it may be emulated via higher-order passive RC or RLC trees, as described in Refs. 1–3 for simulation purposes.^a However, very recently the authors of Ref. 6 have described and demonstrated a single apparatus which preforms as a physical fractional-order capacitor. It is not easy to reconstruct the apparatus described in Ref. 6, but it is an indication that a fractance device might soon be commercially available. Hence, it is important to generalize the filter design theory to the fractional-order domain. This work is a contribution in this direction.

Fractional calculus is the field of mathematics which is concerned with the investigation and application of derivatives and integrals of arbitrary (real or complex) order.^{7–10} The Riemann–Liouville definition of a fractional derivative of order α is given by

$$D^\alpha f(t) := \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau & m-1 < \alpha < m, \\ \frac{d^m}{dt^m} f(t) & \alpha = m. \end{cases} \quad (1)$$

A more physical interpretation of a fractional derivative is given by the Grünwald–Letnikov approximation^b

$$D^\alpha f(t) \triangleq (\Delta t)^{-\alpha} \sum_{j=0}^m \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} f((m-j)\Delta t), \quad (2)$$

where Δt is the integration step. Applying the Laplace transform is widely used to describe electronic circuits in the complex frequency s -domain. Hence, applying the Laplace transform to Eq. (1), assuming zero initial conditions, yields^{7–10}

$$L \{ {}_0d_t^\alpha f(t) \} = s^\alpha F(s), \quad (3)$$

where ${}_0d_t^\alpha f(t) = df(t)/dt$ with zero initial conditions. In this work, we explore the characteristics of a filter which includes a single fractance device. We derive expressions for the filter center frequency and quality factor and also for the half-power and right-phase frequencies. PSpice simulations of three passive fractional-order filters are shown using higher-order emulation trees.^{1–3} Finally, impedance and frequency scaling are discussed in the case of fractional-order filters. It is worth noting that a fractional-order Wien bridge oscillator, which includes two equal-value equal-order fractional capacitors, was studied in Ref. 11.

^aThe use of a high integer-order transfer function to emulate a fractional-order transfer function whose order is less than 1 was also explained in Chap. 3 of Ref. 5. This approximation is based on a Bode-plot approximation but does not imply equivalence in the state space.

^bNumerical simulations in this paper are carried out using a backward difference method based on the Grünwald–Letnikov approximation.

2. Single Fractional Element Filters

The general transfer function of a filter with one fractional element is

$$T(s) = \frac{bs^\beta + d}{s^\alpha + a}. \quad (4)$$

From the stability point of view, this system is stable if and only if $a > 0$ and $\alpha < 2$ while it will oscillate if and only if $a > 0$ and $\alpha = 2$; otherwise it is unstable.¹² The location of the poles is important to determine the filter center frequency ω_o and its quality factor Q . From Eq. (4) it is seen that the poles in the s -plane are located at $s_p = a^{1/\alpha} e^{\pm j(2m+1)\pi/\alpha}$, $m \in I^+$. The possible range of the angle in physical s -plane is $|\theta_s| < \pi$ and hence there are no poles in the physical s -plane for $\alpha < 1$. For $1 < \alpha < 2$, there are only two poles located at $s_{1,2} = a^{1/\alpha} e^{\pm j(\pi/\alpha)}$. Comparing with a classical second-order system whose poles are located $s_{1,2} = (-\omega_o/2Q) \pm j\omega_o\sqrt{1 - (1/4Q^2)} = \omega_o e^{\pm j\delta}$ where $\delta = \cos^{-1}(-1/2Q)$, it can be seen that ω_o and Q are given, respectively, by

$$\omega_o = a^{1/\alpha}, \quad Q = \frac{-1}{2 \cos(\pi/\alpha)}. \quad (5)$$

From the above equation, Q is negative for $\alpha < 1$, and positive for $\alpha \geq 1$. At $\alpha = 1$ (classical first-order filter), it is seen that $Q = 0.5$ as expected. It is important now to define the following critical frequencies

- (1) ω_m is the frequency at which the magnitude response has a maximum or a minimum and is obtained by solving the equation $(d/d\omega)|T(j\omega)|_{\omega=\omega_m} = 0$.
- (2) ω_h is the half-power frequency at which the power drops to half the passband power, i.e., $|T(j\omega_h)| = (1/\sqrt{2})|T(j\omega_{\text{passband}})|$.
- (3) ω_{rp} is the right-phase frequency at which the phase $\angle T(j\omega_{rp}) = \pm \pi/2$.

2.1. Fractional-order low-pass filter

Consider the fractional low-pass filter (FLPF) whose transfer function is

$$T_{\text{FLPF}}(s) = \frac{d}{s^\alpha + a}. \quad (6)$$

The magnitude and phase of this transfer function are

$$\begin{aligned} |T_{\text{FLPF}}(j\omega)| &= \frac{d}{\sqrt{\omega^{2\alpha} + 2a\omega^\alpha \cos(\alpha\pi/2) + a^2}}, \\ \angle T_{\text{FLPF}}(j\omega) &= -\tan^{-1} \frac{\omega^\alpha \sin(\alpha\pi/2)}{\omega^\alpha \cos(\alpha\pi/2) + a}. \end{aligned} \quad (7)$$

The important critical frequencies for this FLPF are found as $\omega_m = \omega_o(-\cos(\alpha\pi/2))^{1/\alpha}$, $\omega_{rp} = \omega_o/(-\cos(\alpha\pi/2))^{1/\alpha}$, and $\omega_h = \omega_o(\sqrt{1 + \cos^2(\alpha\pi/2)} - \cos(\alpha\pi/2))^{1/\alpha}$, where ω_o is as given by Eq. (5). From these expressions it is seen that both ω_m and ω_{rp} exist only if $\alpha > 1$, in agreement with the quality-factor expression (5). Table 1 summarizes the magnitude and phase values at some important

Table 1. Magnitude and phase values at important frequencies for the FLPF.

$\omega =$	$ T_{\text{FLPF}}(j\omega) $	$\angle T_{\text{FLPF}}(j\omega)$
0	$\frac{d}{a}$	0
ω_o	$\frac{d}{2a \cos(\alpha\pi/4)}$	$-\frac{\alpha\pi}{4}$
$\rightarrow \infty$	0	$-\frac{\alpha\pi}{2}$
ω_m	$\frac{d}{a \sin(\alpha\pi/2)}$	$\frac{(1-\alpha)\pi}{2}$
ω_h	$\frac{d}{\sqrt{2}a}$	$\tan^{-1} \frac{\sin(\alpha\pi/2)}{2 \cos(0.5\alpha\pi) + \sqrt{1 + \cos^2(\alpha\pi/2)}}$
ω_{rp}	$\frac{d}{a} \cot(\alpha\pi/2)$	$\frac{\pi}{2}$

frequencies. Figure 1(a) plots the values of ω_m , ω_h , and ω_{rp} (normalized with respect to ω_o) for different values of the fractional-order α . Figure 1(b) is a plot of transfer function magnitude (normalized with respect to $|T(j0)|$) at different critical frequencies versus α . Finally, Fig. 1(c) shows numerically simulated filter magnitude and phase responses for two different cases $\alpha = 0.4$ and $\alpha = 1.6$, respectively. Note the peaking in the magnitude response at $\alpha = 1.6$ which is classically only possible to observe in a second-order ($\alpha = 2$) low-pass filter.

2.2. Fractional-order high-pass filter

Consider the fractional high-pass filter (FHPF) whose transfer function is

$$T_{\text{FHPF}}(s) = \frac{bs^\alpha}{s^\alpha + a}. \quad (8)$$

The magnitude and phase of this transfer function are

$$|T_{\text{FHPF}}(j\omega)| = \frac{b\omega^\alpha}{\sqrt{\omega^{2\alpha} + 2a\omega^\alpha \cos(\alpha\pi/2) + a^2}}, \quad (9)$$

$$\angle T_{\text{FHPF}}(j\omega) = \frac{\alpha\pi}{2} + \angle T_{\text{FLPF}}(j\omega).$$

The important critical frequencies are found as $\omega_m = (-a/\cos(\alpha\pi/2))^{1/\alpha}$, $\omega_{\text{rp}} = (-a \cos(\alpha\pi/2))^{1/\alpha}$, and $\omega_h = \omega_o[\cos(\alpha\pi/2) + \sqrt{1 + \cos^2(\alpha\pi/2)}]^{1/\alpha}$. Figure 2 plots the filter magnitude and phase responses for the two cases $\alpha = 0.4$ and $\alpha = 1.6$.

2.3. Fractional-order band-pass filter

Consider the fractional system

$$T_{\text{FBPF}}(s) = \frac{bs^\beta}{s^\alpha + a}, \quad (10)$$

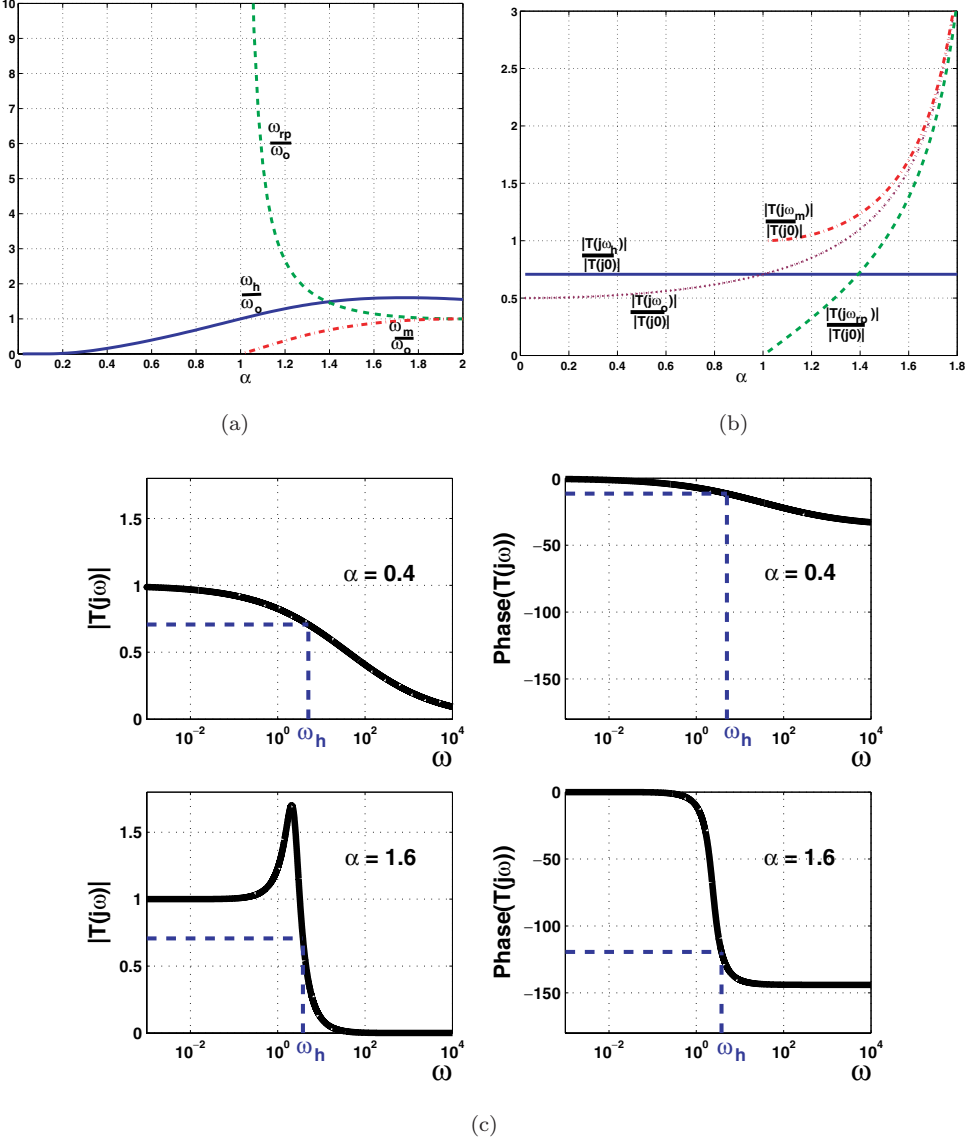


Fig. 1. Numerical simulation for the FLFPF representing (a) normalized critical frequencies versus α , (b) normalized magnitude response versus α , and (c) filter magnitude and phase responses at $\alpha = 0.4$ and $\alpha = 1.6$ assuming $a = d = 4$.

whose magnitude and phase functions are

$$\begin{aligned}
 |T_{\text{FBPF}}(j\omega)| &= \frac{b\omega^\beta}{\sqrt{\omega^{2\alpha} + 2a\omega^\alpha \cos(\alpha\pi/2) + a^2}}, \\
 \angle T_{\text{FBPF}}(j\omega) &= \frac{\pi}{2} - \angle T_{\text{FLPF}}(j\omega).
 \end{aligned} \tag{11}$$

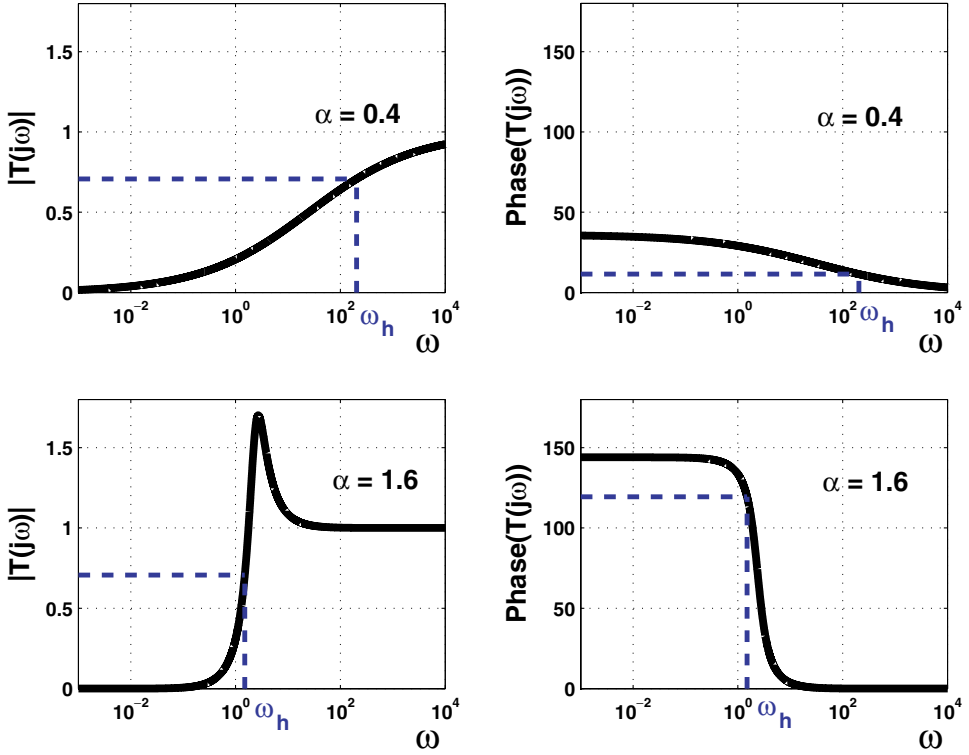


Fig. 2. Magnitude and phase response of the FHPF when $a = 4$ and $b = 1$ for $\alpha = 0.4$ and $\alpha = 1.6$, respectively.

Table 2. Magnitude and phase values at important frequencies for the FBPF.

$\omega =$	$ T_{\text{FBPF}}(j\omega) $	$\angle T_{\text{FBPF}}(j\omega)$
0	0	$\frac{\pi}{2}$
ω_o	$\frac{ba^{\beta/\alpha}}{2a \cos(\alpha\pi/4)}$	$\frac{\beta\pi}{2} - \frac{\alpha\pi}{4}$
$\rightarrow \infty$	$b\omega^{(\beta-\alpha)}$	$\frac{(\beta - \alpha)\pi}{2}$

Table 2 summarizes the magnitude and phase values at important frequencies for this system. It is seen from Table 2 that for $\beta < \alpha \Rightarrow \lim_{\omega \rightarrow \infty} |T(j\omega)| = 0$ and hence the filter can be a band-pass filter (BPF). For $\beta = \alpha \Rightarrow \lim_{\omega \rightarrow \infty} |T(j\omega)| = b$ which makes the filter a HPF. The maxima frequency ω_m is equal to $\omega_o \cdot (X)^{1/\alpha}$ where X is given by

$$X = \frac{\cos(\alpha\pi/2)[(2\beta - \alpha) + \sqrt{\alpha^2 + 4\beta(\alpha - \beta) \tan^2(\alpha\pi/2)}]}{2(\alpha - \beta)}. \quad (12)$$

The case $\alpha = 2\beta$ yields $\omega_m = \omega_o$. It is easy to see that there is always a maximum point in the magnitude response if $\alpha > \beta$. Figure 3(a) shows the value of X versus α for different ratios of β and α . Figure 3(b) shows the magnitude response for the filter when $a = b = 1$. Note from Fig. 3(b) that the center frequency ω_o is not

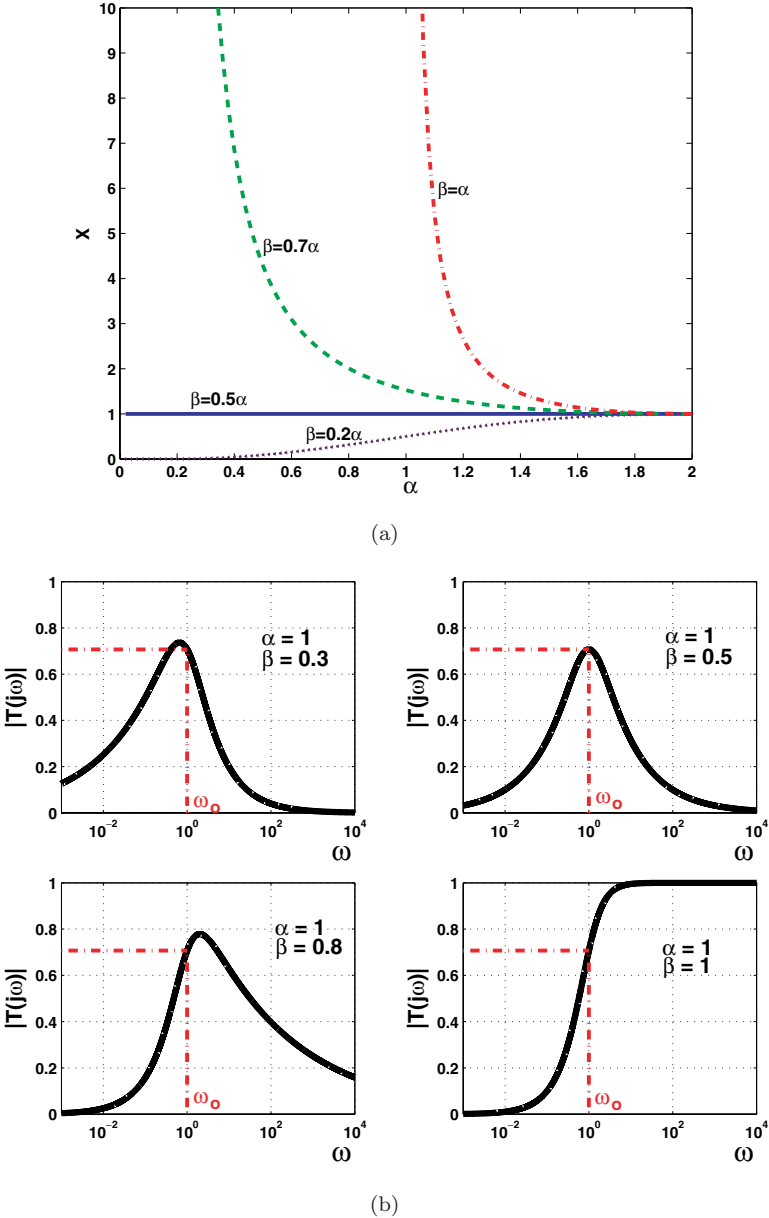


Fig. 3. Numerical simulations for the FBPF (a) values of X versus α and (b) magnitude response for different values of α and β ($a = b = 1$).

necessarily equal to ω_m (where the maxima occurs); which is significantly different from what is known in integer-order filters.

2.4. Fractional-order all-pass filter

Consider the fractional all-pass filter system

$$T_{\text{APF}}(s) = \frac{b(s^\alpha - a)}{s^\alpha + a}. \quad (13)$$

The magnitude and phase of this system are, respectively,

$$\begin{aligned} |T_{\text{FAPF}}(j\omega)| &= b \frac{\sqrt{\omega^{2\alpha} - 2a\omega^\alpha \cos(\alpha\pi/2) + a^2}}{\sqrt{\omega^{2\alpha} + 2a\omega^\alpha \cos(\alpha\pi/2) + a^2}}, \\ \angle T_{\text{FAPF}}(j\omega) &= \angle b \frac{(\omega^\alpha \cos(\alpha\pi/2) - a) + j\omega^\alpha \sin(\alpha\pi/2)}{(\omega^\alpha \cos(\alpha\pi/2) + a) + j\omega^\alpha \sin(\alpha\pi/2)}. \end{aligned} \quad (14)$$

Table 3 summarizes the magnitude and phase values at important frequencies for this filter. It is seen here that $\omega_m = \omega_{\text{rp}} = \omega_o$ and that at this frequency a minima occurs if $\alpha < 1$ and a maxima occurs if $\alpha > 1$ while the magnitude remains flat when $\alpha = 1$ (classical integer-order all-pass filter). The half-power frequency is given by $\omega_h = \omega_o [2 \cos(\alpha\pi/2) + \sqrt{4 \cos^2(\alpha\pi/2) - 1}]^{1/\alpha}$. Figure 4 shows the magnitude and phase responses for the two different cases $\alpha = 0.4$ and $\alpha = 1.6$.

3. PSpice Simulations

Passive filters are chosen for simulations. In all simulations, we fix the fractance device to a fractional capacitor of order $\alpha = 0.4, 1$, or 1.6 . Figure 5(a) is the structure proposed in Ref. 1 to simulate a fractional capacitor of order 0.5 ($Y_{C_F} = C_F s^{0.5}$; $C_F = \sqrt{C/R}$) while the circuit in Fig. 5(b), proposed in Ref. 3, is used to simulate a fractional capacitor of arbitrary order α ($Y_{C_F} = C_F s^\alpha$). To realize $\alpha = 0.4$, for example, we need $n = 31$ branches with $R_{n+1}/R_n \simeq 0.5686$ and $C_{n+1}/C_n \simeq 0.4287$.³ To simulate a capacitor of order $\alpha = 1.6$, a floating GIC circuit¹³ is used. The input impedance of a GIC is $Z_i = Z_1 Z_2 Z_3 / Z_4 Z_5$; taking $Z_3 = Z_4 = R$, $Z_1 = Z_2 = 1/sC_F$, and $Z_5 = 1/s^{0.4}C_F$ results in $Z_i = 1/s^{1.6}C_F$.

Figures 5(c) and 5(d) show a passive FLPF and PSpice simulation of its magnitude response in three cases $\alpha = 0.4, 1$, and 1.6 , respectively. Figures 5(e) and 5(f) show a passive FHFPF and its PSpice magnitude response while Figs. 5(g) and 5(h)

Table 3. Magnitude and phase values at important frequencies for the FAPF.

$\omega =$	$ T_{\text{FAPF}}(j\omega) $	$\angle T_{\text{FAPF}}(j\omega)$
0	b	π
ω_o	$b \tan\left(\frac{\alpha\pi}{4}\right)$	$\frac{\pi}{2}$
$\rightarrow \infty$	b	0

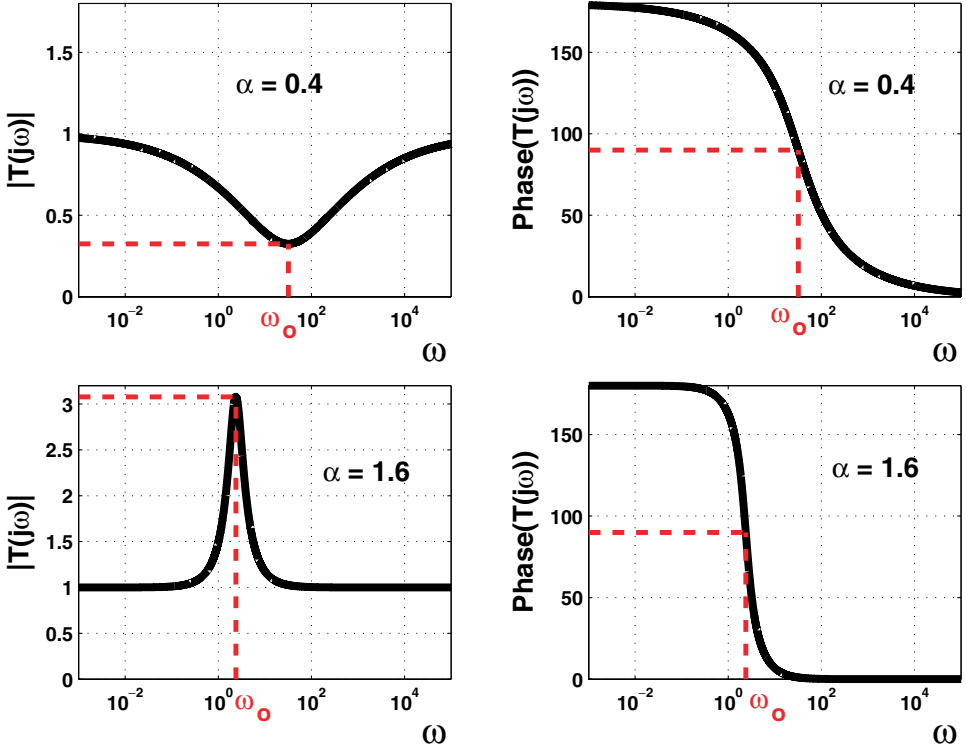


Fig. 4. FAPF magnitude and phase responses when $\alpha = 0.4$ and $\alpha = 1.6$ ($a = 4, b = 1$).

show a passive FAPF and its PSpice simulation results for the same three values of α . Note the peaking in the response for $\alpha > 1$ which is expected to increase as the filter approaches $\alpha = 2$, i.e., second-order filter.

We have constructed the fractional low-pass filter of Fig. 5(c) in the laboratory using one of the fractional capacitors donated by the authors of Ref. 6. This fractional capacitor has $\alpha \approx 0.5$ up to approximately 30 kHz. Results are shown in Fig. 6 compared with a normal capacitor.

4. Scaling

Impedance and frequency scaling can be used to adjust the filter component values or operating frequency. A fractional-order filter is similar to an integer-order one in terms of impedance scaling. However, for frequency scaling and assuming all critical frequencies are to be scaled by a factor λ in which the new frequencies equal λ times the old ones, then the components may be scaled according to the following equations

$$R_{\text{new}} = \frac{1}{\lambda^\alpha} R \quad \text{or} \quad C_{\text{new}} = \frac{1}{\lambda^\alpha} C_{\text{old}}. \quad (15)$$

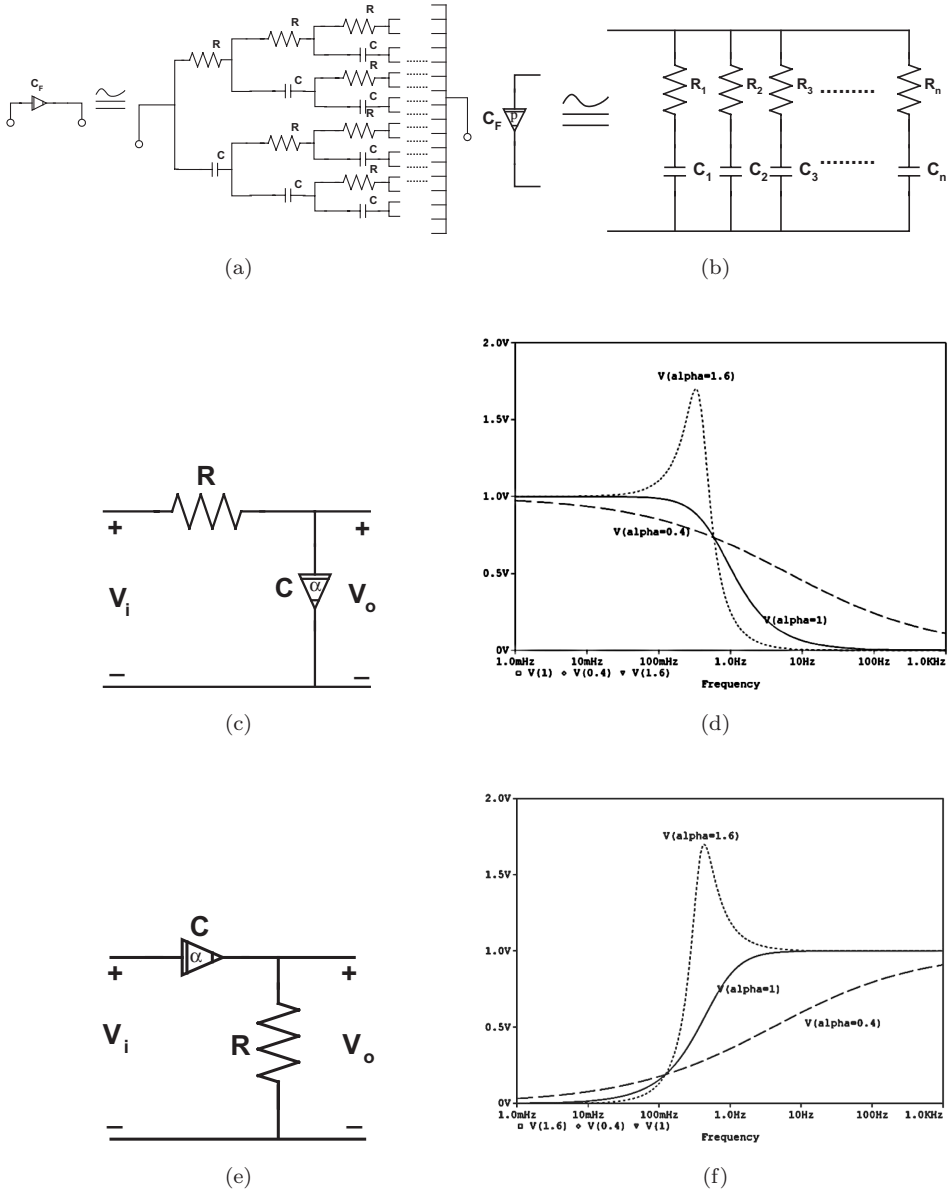


Fig. 5. (a) Realization of a fractional capacitor of order 0.5,¹ (b) realization of a fractional capacitor of order $\alpha < 1$,³ (c) FLPF circuit with transfer function $T(s) = d/(s^\alpha + a)$, $d = a = 1/RC = 4$, (d) PSpice simulation of the FLPF ($R = 6.74 \text{ k}\Omega$, $C = C_F = 37 \mu\text{F}$), (e) FHPF circuit with transfer function $T(s) = ds^\alpha/(s^\alpha + a)$, $d = 1$, $a = 1/RC = 4$, (f) PSpice simulation of the FHPF ($R = 6.74 \text{ k}\Omega$, $C = C_F = 37 \mu\text{F}$), (g) FAPF circuit with transfer function $T(s) = -(1/2)(s^\alpha - a)/(s^\alpha + a)$, $a = 1/RC = 4$, and (h) PSpice simulation of the FAPF ($R = 6.74 \text{ k}\Omega$, $C = C_F = 37 \mu\text{F}$).

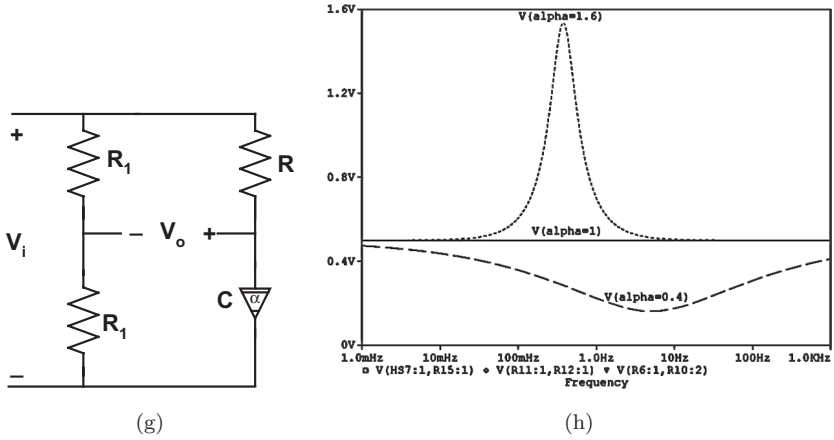


Fig. 5. (Continued)

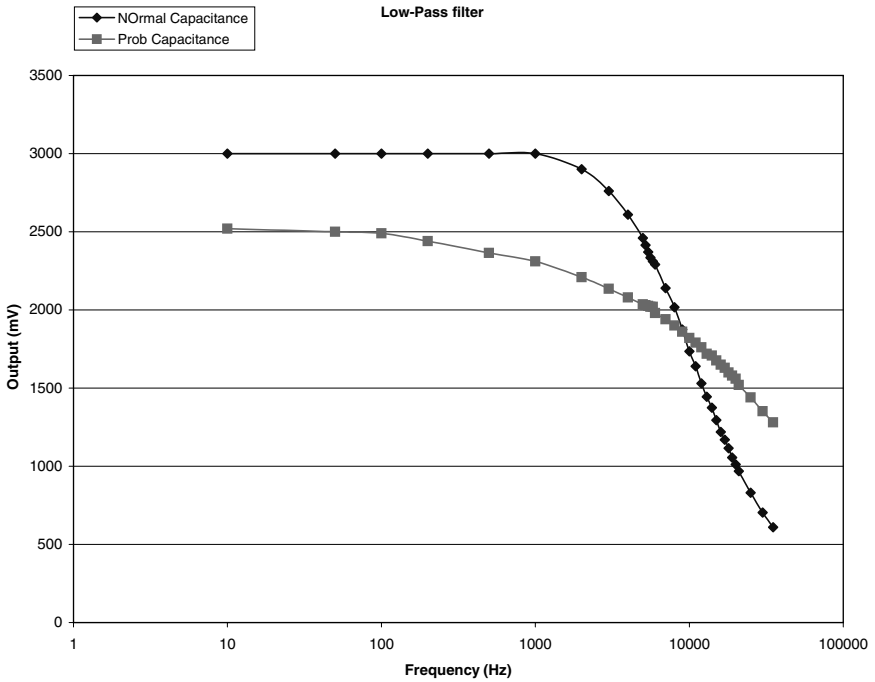


Fig. 6. Experimental results of a fractional-order low-pass filter with $\alpha \approx 0.5$ compared with a normal capacitor.

5. Conclusion

In this work, we have generalized classical first-order filter networks to be of fractional-order. We have shown simulation results for filters of order $0 < \alpha < 1$

and $1 < \alpha < 2$. It is clear that more flexibility in shaping the filter response can be obtained via a fractional-order filter. It is also clear that the band-pass filter, classically known to be realizable only through a second-order system, can actually be realized by a fractional filter of order $1 < \alpha < 2$.

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