Second Order Approximation of the Fractional Laplacian Operator for Equal-Ripple Response

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***Abstract*—In this paper we propose a modification to a second order approximation of the fractional-order Laplacian operator, s^{α}, where $0 < \alpha < 1$. We show how this proposed modification can be used to change the ripple error of both the magnitude and phase responses of the approximation when compared to the ideal case. Equal-ripple magnitude and phase responses that have both less cumulative error and less maximum ripple deviation are presented using this modification. A 1^{st} order lowpass filter with fractional step of 0.8, that is of order 1.8, is implemented using the proposed approximation. Experimental results verify the operation of this approximation in the realization of the fractional step filter.**

I. Introduction

Traditionally the Laplacian operator, s, is raised to an integer order, i.e $s, s^2, ...s^n$, when used in the design of analog circuits. However, it is also mathematically valid to raise to a non-integer order, s^{α} , where $0 < \alpha < 1$; representing a fractional order system. A fractional derivative may be defined, according to the Riemann-Liouville definition [1], as

$$\frac{d^{\alpha}}{dt^{\alpha}}f(t) \equiv D^{\alpha}f(t) = \frac{d}{dt}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}}d\tau\right] \quad (1)$$

where $\Gamma(\cdot)$ is the gamma function. Applying the Laplace transform, with zero initial conditions, to (1) yields

$$L\left\{{}_0d_t^{\alpha}f(t)\right\} = s^{\alpha}F(s) \quad (2)$$

This fractional Laplacian operator has practical applications in numerous areas of engineering [2]. However, at this time there are no commercial fractance devices available. Therefore, to physically realize circuits that make use of the advantages of s^{α}, integer order approximations have been used. There are many methods to create an approximation of s^{α} that include Continued Fraction Expansions (CFEs) as well as numerous rational methods [3]. These methods present a large array of approximations with varying order and accuracy, with the accuracy and approximated frequency band increasing as the order of the approximation increases. The importing of these concepts into circuit theory is relatively new, and has shown applications in power electronics [4], integrator [5], [6] and differentiator circuits [7], multivibrator circuits [8], and filter theory [9], [10] with potentially many other applications.

In this paper, we propose a modification to a second-order approximation for the fractional Laplacian, s^{α}, which can be used to manipulate the error of both the magnitude and phase response. We show how this approximation can be manipulated to create an equi-ripple error response when compared to the ideal case. A fractional lowpass filter of order 1.8 is verified experimentally to show the application of this approximation.

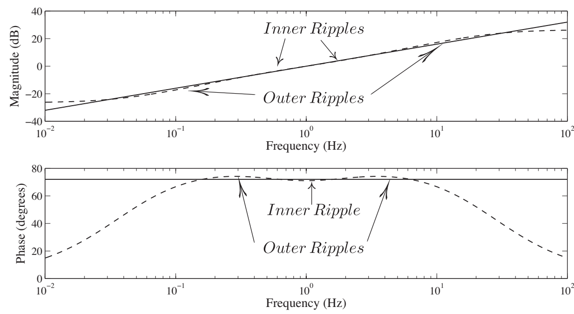

Figure 1. Magnitude and phase of ideal (solid) and 2nd order approximation (dashed) of s^{α} when $\alpha = 0.8$

II. Laplacian Operator Approximation

Previous work on the approximation of the fractional Laplacian operator has yielded a second order approximation using the CFE method [11] as,

$$s^{\alpha} \approx \frac{(\alpha^2+3\alpha+2)s^2+(8-2\alpha^2)s+(\alpha^2-3\alpha+2)}{(\alpha^2-3\alpha+2)s^2+(8-2\alpha^2)s+(\alpha^2+3\alpha+2)} \quad (3)$$

The magnitude and phase response of (3) compared with s^{α} when $\alpha = 0.8$ is shown in Fig. 1. From this figure, the magnitude error of the approximation compared to the ideal case does not exceed $1.338\,dB$ for $\omega\,\epsilon\,[0.023, 42.85]\,rad/s$; while the phase error does not exceed 2.173 for $\omega\,\epsilon\,[0.127, 7.87]\,rad/s$.

We note also both the magnitude and phase responses have inner and outer ripples, as highlighted in Fig. 1. To compare the peaks of the inner and outer ripples for both the magnitude and phase, the error of (3) compared to the ideal s^{α} was calculated as

$$Error = s^{\alpha} - \frac{(\alpha^2+3\alpha+2)s^2+(8-2\alpha^2)s+(\alpha^2-3\alpha+2)}{(\alpha^2-3\alpha+2)s^2+(8-2\alpha^2)s+(\alpha^2+3\alpha+2)} \quad (4)$$

Using (4) the peaks of the magnitude and phase ripple errors were calculated numerically as shown in Fig. 2, for $0 < \alpha < 1$, illustrating that both the inner and outer ripple errors vary significantly with α. Note the size of the outer ripple is

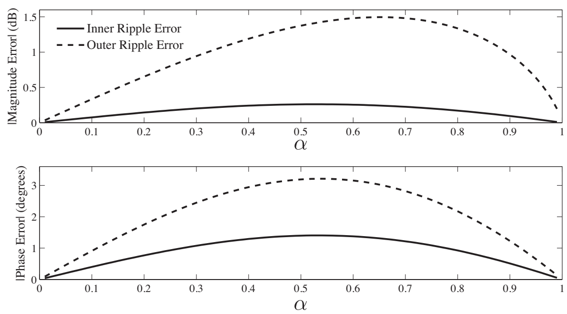

Figure 2. Absolute error of inner and outer ripples vs. α of approximated s^{α} compared to the ideal of both magnitude and phase responses

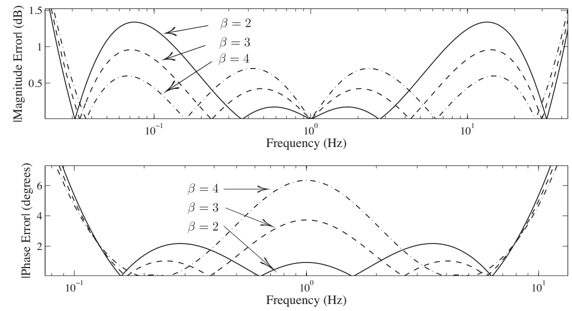

Figure 3. (a) Magnitude and phase absolute inner and outer ripple error of (5) compared to the ideal s^{α} for β values from 2 to 4 when $\alpha = 0.8$

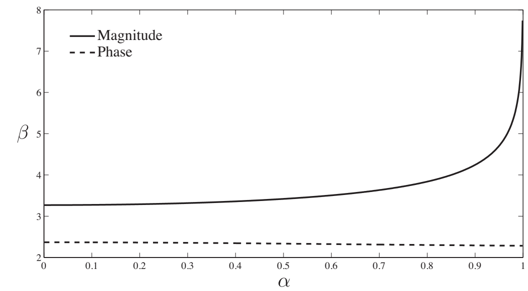

Figure 4. β values required for equi-ripple magnitude (solid) and phase (dashed) error approximation for $0 < \alpha < 1$

always larger than the inner ripple for all values of α and the maximum error for magnitude and phase occurs at different values of α. The magnitude error reaching a maximum of $1.496dB$ when $\alpha = 0.65$ with an inner ripple of $0.243dB$, and the phase error reaching a maximum error of $3.211°$ when $\alpha = 0.54$ with an inner ripple of $1.404°$.

III. Modified Approximation For Ripple Manipulation

Building on the 2nd order approximation of [11], we propose the following modification of the approximation of the fractional Laplacian operator. That is,

$$s^{\alpha} \approx \frac{a_0 s^2 + a_1 s + a_2}{a_2 s^2 + a_1 s + a_0} \tag{5}$$

where $a_0 = (\alpha^2 + 3\alpha + 2)$, $a_1 = (\beta + 6 - \beta\alpha^2)$, $a_2 = (\alpha^2 - 3\alpha + 2)$, $0 < \alpha < 1$ and $\beta \,\epsilon\, \Re$. Introducing β allows for the manipulation of the ripple error in both the magnitude and phase responses. Compared to (3) only a_1 is being modified. While the previous approximation (3) was stable for all values of α when $0 < \alpha < 1$, the introduction of β limits the range of values that realize a stable approximation. To satisfy this stability requirement, the value of β in the approximation must meet the criteria

$$\beta > \frac{-6}{1 - \alpha^2} \tag{6}$$

In the following section we examine how this modification can be used to manipulate the error of the response and evaluate the cumulative error.

A. Error Ripple Manipulation

Using (5) the size of both the inner and outer error ripples in both the magnitude and phase responses can be changed by modifying the value of β. Increasing β decreases the size of the outer ripple error while increasing that of the inner ripple error in both responses. Considering the case when $\beta = 2$ in Fig. 3, when the modified approximation (5) is equal to the original approximation (3), we see that the outer ripple error of the magnitude response is $1.481dB$ with an inner ripple error of $0.225dB$; while error ripples of the phase response are $2.818°$ and $1.217°$ for the outer and inner error ripples, respectively. Note that increasing β to 3 and 4 increases the inner ripple of the magnitude response while also decreasing the outer ripple. This same behaviour is mirrored in the phase response and is shown in Fig. 3. With the ability to increase the size of the inner ripple while decreasing that of the outer ripple, there exists a value of β that will yield an equi-ripple error response for either phase or magnitude for a given α. That is, a response where both the inner and outer ripple errors are the same size. These values of β that yield this equi-ripple response, for both magnitude and phase, using (5) are shown in Fig. 4 for $0 < \alpha < 1$. Note that all the values of β that create an equi-ripple response are positive meeting the stability criteria of (6) and will result in a stable approximation. Also, from this data there is no intersection of the values for magnitude and phase equi-ripple response, so that it is never possible to have an approximation with equi-ripple response in both magnitude and phase simultaneously. This equi-ripple response is shown in Fig. 5 for the cases when $\alpha = 0.2$ and $\alpha = 0.8$ using $\beta = 3.2907$ and $\beta = 3.8382$, respectively. Comparing this ripple error to that in Fig. 2, the error of the equi-ripple response is lower than the maximum ripple error. For the case when $\alpha = 0.8$, the max ripple errors are $1.337dB$ and $2.173°$ for the magnitude and phase, respectively. Using the equi-ripple β value in (4) shows a reduction of the max error to values of $0.6542dB$ and $1.794°$. That is a reduction of $0.6828dB$ and $0.379°$ in the ripple error of the magnitude and phase responses, respectively. This is also true for the case when $\alpha = 0.2$ with a reduction of $0.2809dB$ and $0.266°$.

B. Cumulative Error

In order to quantify the error of both approximations to the ideal case, we use an objective criteria [6] to compare them,

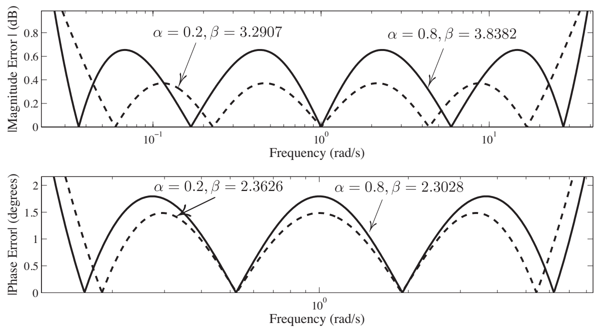

Figure 5. Equiripple magnitude and phase responses for the cases when $\alpha = 0.2$ and $\alpha = 0.8$.

α	EquiRipple Error		Original Error	
	Mag	Phase	Mag	Phase
0.1	75.11	6523	77.59	6718
0.3	604.0	55779	624.0	57347
0.5	1274	136650	1317	140049
0.7	1330	200412	1375	204462
0.9	213.1	113251	233.7	114666

Table I
CUMULATIVE ERROR OF BOTH MAGNITUDE AND PHASE EQUI-RIPPLE RESPONSES COMPARED TO THE ORIGINAL APPROXIMATION.

defined as

$$Cumulative\,Error = \sum_{i=1}^{N}(X_i - Y_i)^2 \tag{7}$$

where X_i is the approximation value of either phase or magnitude for a given frequency, Y_i is the ideal value for the same frequency value, and N is the number of samples for the comparison. In our example we chose $N = 300$ over a bandwidth of $\omega \,\epsilon\, [0.01, 100] rad/s$. The MATLAB simulation results for selected values of α, where the value of β used is selected from Fig. 4, are shown in Table I. From these results it can be seen that the cumulative error of both magnitude and phase of the equi-ripple response is less than the approximation of (3). So while the modification may increase the inner ripple error of the approximation, it still reduces the total overall error of approximation when compared to the ideal s^α.

IV. APPLICATION TO FRACTIONAL FILTERS

The use of s^α in filter theory has the potential to create a more general filter design methodology. Previously the stopband attenuation was limited to an integer order step, but with the use of s^α this expands to a fractional order step [9], [10]. However, the current implementation of these fractional-step filters requires the use of an approximation to s^α [12]. Using (5) in the realization of these approximated fractional step filters is a more suitable choice than (3) as it has both lower cumulative and lower peak errors. In this section we present the design of a fractional step filter using (5) and compare it and a fractional step filter using (3) against the ideal filter.

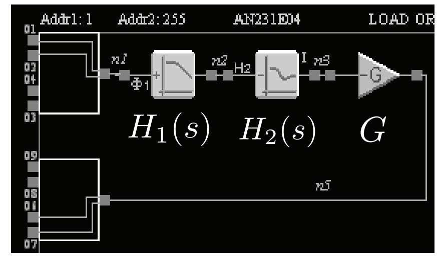

Figure 6. FLPF implementation using the bilinear filter, biquadratic filter, and inverting gain CAMs of the AnadigmDesigner2 tools.

A. Fractional Low Pass Filter (FLPF)

To realize a FLPF (5) is substituted into the fractional transfer function given in [12] as

$$H^{LP}_{1+\alpha}(s) = \frac{1}{s^\alpha(s + k_2) + k_3} \tag{8}$$

which then becomes,

$$H^{LP}_{1+\alpha}(s) \backsimeq \frac{\left(\frac{a_2}{a_0}s^2 + \frac{a_1}{a_0}s + 1\right)}{s^3 + c_0 s^2 + c_1 s + c_2} \tag{9}$$

where $c_0 = (a_1 + a_0 k_2 + a_2 k_3)/a_0$, $c_1 = (a_1(k_2 + k_3) + a_2)/a_0$, $c_2 = (a_0 k_3 + a_2 k_2)/a_0$, α is the fractional step and $k_{2,3}$ are values from Fig. 6 in [12] selected to maintain a passband similar to the Butterworth response. In order to verify (9) a Field Programmable Analog Array (FPAA) provides a useful test bed by virtue of the fact that it can be easily configured into bilinear and biquadratic blocks. Using the AnadigmDesigner2 design tools the bilinear and biquadratic Configurable Analog Modules (CAMs) were used to realize (9) on a Anadigm AN231E04 FPAA with the transfer function arranged to match the CAMs, taking the form

$$\begin{aligned} H(s) &= G \cdot H_1(s) \cdot H_2(s) \\ H_1(s) &= \frac{2\pi f_1 G_1}{s + 2\pi f_1} \\ H_2(s) &= -G_2 \frac{s^2 + \frac{2\pi f_{2_z}}{Q_{2_z}} s + 4\pi^2 f_{2_z}^2}{s^2 + \frac{2\pi f_{2_p}}{Q_{2_p}} s + 4\pi^2 f_{2_p}^2} \end{aligned}$$

where $H_1(s)$, $H_2(s)$, and G are the transfer functions of the bilinear filter, biquadratic filter, and inverting gain CAMs, respectively. The inverting gain CAM is included to achieve the low gain not possible using the biquadratic and bilinear CAMs which are comprised of Switched-Capacitor blocks. Here $G_{1,2}$ is the internal gain of $H_1(s)$ and $H_s(s)$, G is the external gain, f_1 the pole frequency of $H_1(s)$, $f_{2_{p,z}}$ the pole and zero frequencies of $H_2(s)$, and $Q_{2_{p,z}}$ the pole and zero quality factors of $H_2(s)$. The connection of these modules in the AnadigmDesigner2 design environment is shown in Fig. 6.

B. Simulated and Experimental Results

To verify the use of the proposed approximation in the realization of FLPFs, we compare the measured magnitude response of 1.8 order FLPFs using (3) and (5) to the MATLAB

Design Value	Order $(1+0.8)$	
	Original	Modified
$f_1(kHz)$	0.611	0.504
$f_{2_p}(kHz)$	1.30	1.42
$f_{2_z}(kHz)$	4.61	4.60
Q_{2_p}	0.674	0.672
Q_{2_z}	0.163	0.149
$G_{1,2}$	1	1
G	0.0795	0.0963

Table II
REALIZED CAM VALUES FOR PHYSICAL IMPLEMENTATION OF THE 1.8 ORDER FLPFS REALIZED USING THE ORIGINAL AND EQUI-RIPPLE APPROXIMATION OF (3) AND (5), RESPECTIVELY. NOTE $\beta = 3.8382$ FOR THE EQUI-RIPPLE MAGNITUDE APPROXIMATION.

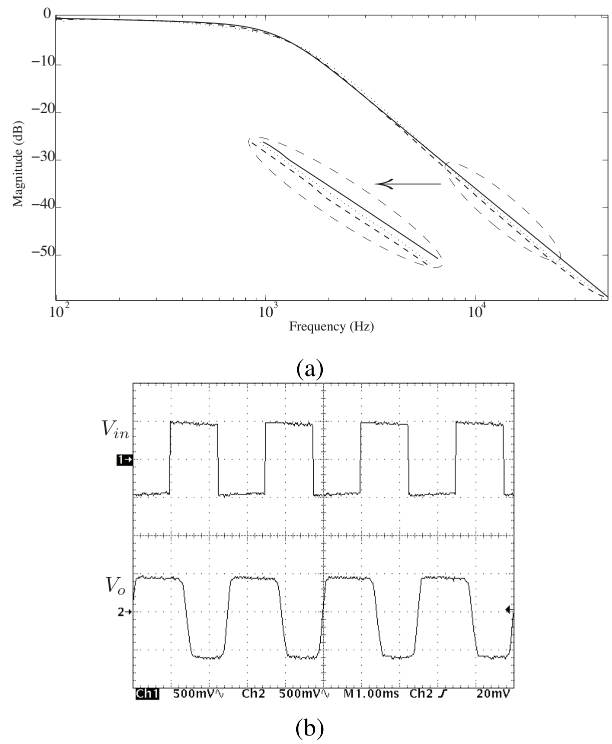

Figure 7. (a) Comparison of the magnitude responses of the simulated ideal (solid) and experimental approximated 1.8 order FLPFs using original (dashed) and modified (dotted) approximations, (3) and (5), respectively. (b) Step response of approximated 1.8 order FLPF using (5).

simulated ideal transfer function of (8). These approximated FLPFs were shifted to a frequency of $1kHz$ and realized using an FPAA configured as shown in Fig. 6, with the design values to realize both approximations detailed in Table II. The magnitude responses of these filters were measured using a HP4395A Network analyzer and are shown in Fig. 7(a) as dashed lines and dotted lines for the FLPFs approximated using (3) and (5), respectively. Comparing these approximated FLPFs to the ideal, shown as a solid line in Fig. 7(a), both responses closely match the passband of the filter. Viewing the stopband we see the differences between implementing the FLPFs using the different approximations of s^α. Note that from approximately $1.6kHz$ to $4.2kHz$ using the equi-ripple approximation results in a larger deviation compared to the ideal than the original approximation, which is expected because the inner ripple error of the approximation increases using the equi-ripple response compared to the original. However, from approximately $4.2kHz$ to $42kHz$ the modified approximation is a closer approximation of (8) than the original modification, highlighted in the inset of Fig. 7(a), a result of the reduced outer ripple error. Therefore, while using (5) increases the error of the approximated FLPF over a small range it also decreases the error over a much larger range in the stopband. This illustrates that the use of the equi-ripple approximation over the original approximation is a better choice in the implementation of FLPFs. The step response of the 1.8 order filter was also investigated to confirm its stability and is shown in Fig. 7(b).

V. CONCLUSION

We have proposed a modified second-order approximation to the fractional Laplacian operator, s^α, that allows for the manipulation of the error ripples in both the magnitude and phase responses. Using this modification technique, an equi-ripple error response for either magnitude or phase has been demonstrated. These equi-ripple responses have both a lower peak error and cumulative error than the original approximation. This lower error makes our modified approximation more attractive than the original approximation for applications where s^α is required. We have implemented fractional step filters to demonstrate the application of this approximation in the realization of electronic circuits for signal processing.

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