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GENERIC RC REALIZATIONS OF CHUA'S CIRCUIT

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We propose novel generic RC realizations of Chua's circuit. These realizations are based on the simplest possible models for second-order RC sinusoidal oscillators that are used to replace the active tank resonator in the classical Chua's circuit configuration. The sinusoidal oscillators are represented by circuit-independent *black-box* models. Hence, numerous circuit realizations can be derived.

1. Introduction

Since its discovery, Chua's circuit has served as the main prototype circuit for studying chaos in electronic systems [Matsumoto, 1984]. Several realizations of this circuit have been introduced in the literature [Cruz & Chua, 1992; Kennedy, 1992, 1995; Rodriguez-Vazquez & Delgado-Restituto, 1993; Zhong, 1994; Morgul, 1995; Elwakil & Kennedy, 1999a, 1999b, 2000]. These various implementations were concerned with two main design aspects of Chua's circuit. The first aspect is the realization of the voltage-controlled nonlinear resistor [Cruz & Chua, 1992; Kennedy, 1992; Zhong, 1994; Elwakil & Kennedy, 1999b, 2000] and the second aspect is the removal of the inductor as it is generally difficult to integrate [Rodriguez-Vazquez & Delgado-Restituto, 1993; Morgul, 1995; Elwakil & Kennedy, 1999a]. We have contributed to both of these research topics. In the case of the voltage-controlled nonlinear resistor, we have proposed a simple high performance realization based on the current-feedback op amp [Elwakil & Kennedy, 2000]. We then replaced this active nonlinearity by a simple diode [Elwakil & Kennedy, 1999b]. Of course, with the diode one cannot produce double-scroll dynamics, but can produce chaos with qualitative dynamics similar to those of the chaotic Colpitts oscillator [Kennedy, 1995]. Concerning the removal of the inductor, our approach was significantly different from the previous attempts [Rodriguez-Vazquez & Delgado-Restituto, 1993; Morgul, 1995] which were basically replacing the inductor by an RC circuit that emulates its function. Our approach was based on the conjecture proposed in [Elwakil & Kennedy, 1999a] which suggested that at the heart of any autonomous chaotic oscillator, there lies a core sinusoidal oscillator engine. Thus, instead of replacing the inductor alone one can in fact replace the entire active tank resonator (also known as the negative-resistance oscillator) which is the core engine of Chua's circuit. Replacing this engine with any RC sinusoidal oscillator will automatically result in an inductorless Chua's circuit. The significance of this approach lies in its ability to isolate the mechanism by which oscillations are produced. Hence, important features of the chaotic oscillator, such as its spectral bandwidth and the relationship between the values of particular circuit components, can be derived [Elwakil & Kennedy, 1999a]. Furthermore, oscillators based on different nonlinear resistor characteristics can be freely investigated.

In this work, we propose generic RC realizations of Chua's circuit which are circuit-independent. We base our design procedure on the simplest possible models of two popular classes of second-order RC sinusoidal oscillators.

2. Sinusoidal Oscillators

A general second-order RC sinusoidal oscillator has the following state-space representation

$$\begin{bmatrix} \dot{V}_{C1} \\ \dot{V}_{C2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \end{bmatrix} \quad (1)$$

where V_{C1} and V_{C2} are the voltages across its two capacitors respectively. The condition for oscillation and the frequency of oscillation are given respectively by

$$a_{11} + a_{22} = 0 \quad \text{and} \quad \omega_o = \sqrt{a_{11}a_{22} - a_{12}a_{21}} \quad (2)$$

Consider the class of sinusoidal oscillators shown in Fig. 1(a). This class is characterized by having a separate parallel R_1C_1 network, which is usually known as the timing network, and is used to adjust the frequency of oscillation. The current I which supplies the timing network [see Fig. 1(a)] can generally be expressed as

$$I_{\pm} = \pm g_1 V_{C1} \mp g_2 V_{C2} \quad (3)$$

where g_1 and g_2 are constant transconductances.

Since the oscillator is active, either g_1 or g_2 must be negative. Noting that the frequency of oscillation ω_o is generally equal to $\sqrt{n}/R_1C_1$, where n is a multiplication factor, and by applying the conditions of (2), the state equations describing Fig. 1(a) can be written in matrix form as

$$\begin{bmatrix} \dot{V}_{C1} \\ \dot{V}_{C2} \end{bmatrix} = \frac{1}{C} \begin{bmatrix} \pm g_1 - g & \mp g_2 \\ \dfrac{ng^2 + (\pm g_1 - g)^2}{\pm g_2} & g \mp g_1 \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \end{bmatrix} \quad (4)$$

where $C_1 = C_2 = C$ and $g = 1/R_1$.

From (4) it can be seen that the condition of oscillation is satisfied and the frequency of oscillation is given by $\omega_o = \sqrt{n}g/C$. By introducing the dimensionless variables: $\tau = t(g_2/C)$, $X = V_{C_1}/V_{\text{ref}}$, $Y = V_{C2}/V_{\text{ref}}$, $K_1 = g_1/g_2$ and $K_2 = g/g_2$, where V_{ref} is an arbitrary voltage normalization constant, the dimensionless form of (4) (apart from the term ε_c) becomes

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} \pm K_1 - K_2 \pm \varepsilon_c & \mp 1 \\ \pm[nK_2^2 + (\pm K_1 - K_2)^2] & K_2 \mp K_1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \quad (5)$$

Practical oscillators need to have a control parameter to compensate for any losses that may result in the condition for oscillation not to be satisfied and thereby to guarantee that oscillations start. This control parameter is represented in (5) by the small error factor ε_c. However, once oscillations start, an amplitude control mechanism is needed to stabilize the amplitude of oscillation. This mechanism can be a nonlinear voltage-controlled device (usually an FET) inserted in the feedback path, or it can simply be the saturation-type nonlinearity of the active devices employed (e.g. op amps). Thus, the above set of equations cannot faithfully model the behavior of a sinusoidal oscillator since they are linear. However, they do correctly model the function performed by the sinusoidal oscillator within the chaotic oscillator structure since the generation of chaos is not associated with any nonlinear amplitude control mechanism of the sinusoidal oscillator; rather it is a result of the characteristics of another nonlinear composite which is linked to this oscillator.

Next we consider the class of sinusoidal oscillators shown in Fig. 1(b). This class has a separate series R_1C_1 network and is described in state-space form by

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} \pm K_1 \pm \varepsilon_c & \mp 1 \\ \pm[nK_2^2 + K_1^2] & \mp K_1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \quad (6)$$

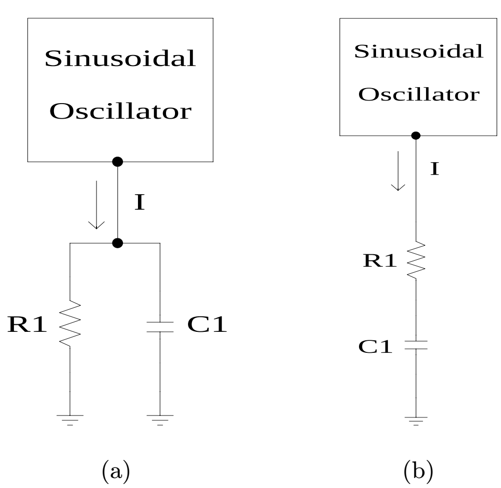

Fig. 1. Class of sinusoidal oscillators with (a) a parallel RC network, (b) a series RC network.

Here, the same dimensionless variables used to obtain (5) were also adopted. Note that K_2 appears in (6) as a frequency multiplication factor and does not affect the condition for oscillation.

In the following section we show how Chua's circuit can be constructed using these two generic classes of sinusoidal oscillators. In particular, the extension of the two-dimensional systems represented by Eqs. (5) and (6) into a three-dimensional space will become clear. We emphasize that Eqs. (5) and (6) describe the dynamics of the sinusoidal oscillators in the simplest possible form.

3. Generic RC Chua's Circuit

Consider the configuration shown in Fig. 2(a) where the generic sinusoidal oscillator of Fig. 1(a) has been substituted for the tank in Chua's circuit. The classical implementation of the voltage-controlled nonlinear resistor known as Chua's diode [Kennedy, 1992] is based on adding two negative resistors in parallel, one of which is linear while the other has the saturation-type characteristics sketched in Fig. 2(a). The linear negative resistor combined with the LC tank circuit form the well-known negative-resistance oscillator which we replace with the generic structure of Fig. 1(a). The remaining parts of the circuit include the coupling resistor R, the switching capacitor C_3 and the nonlinear part of Chua's diode, which has the characteristic shown.

With the same normalization used to derive Eqs. (5) and (6) and in addition to $Z = V_{C3}/V_{\text{ref}}$, $\varepsilon = C_3/C$ and $K = 1/g_2R$, the generic RC Chua's circuit of Fig. 2(a) is described by

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \varepsilon\dot{Z} \end{bmatrix} = \begin{bmatrix} K_1 - K_2 - K \pm \varepsilon_c & -1 & K \\ nK_2^2 + (K_1 - K_2)^2 & K_2 - K_1 & 0 \\ K & 0 & -K - a \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -b \end{bmatrix} \tag{7a}$$

where

$$\begin{cases} a = 0,\ b = A & Z \leq -1 \\ a = -A,\ b = 0 & -1 < Z < 1 \\ a = 0,\ b = -A & Z \geq 1 \end{cases} \tag{7b}$$

Here, we have chosen $I = I_+$ and $V_{\text{ref}} = V_{BP}$, where V_{BP} is the breakpoint voltage of the nonlinear resistor. A is an arbitrary constant obtained directly from the saturation-type characteristics of the nonlinear resistor. By comparing (5) and (7), it can be clearly recognized that the dynamics in the X–Y plane are governed by the sinusoidal oscillator. In particular, (5) is the limiting case of (7) when $K \to 0$ and $\varepsilon \to 0$.

In a similar manner, a generic Chua's circuit can be constructed based on the sinusoidal

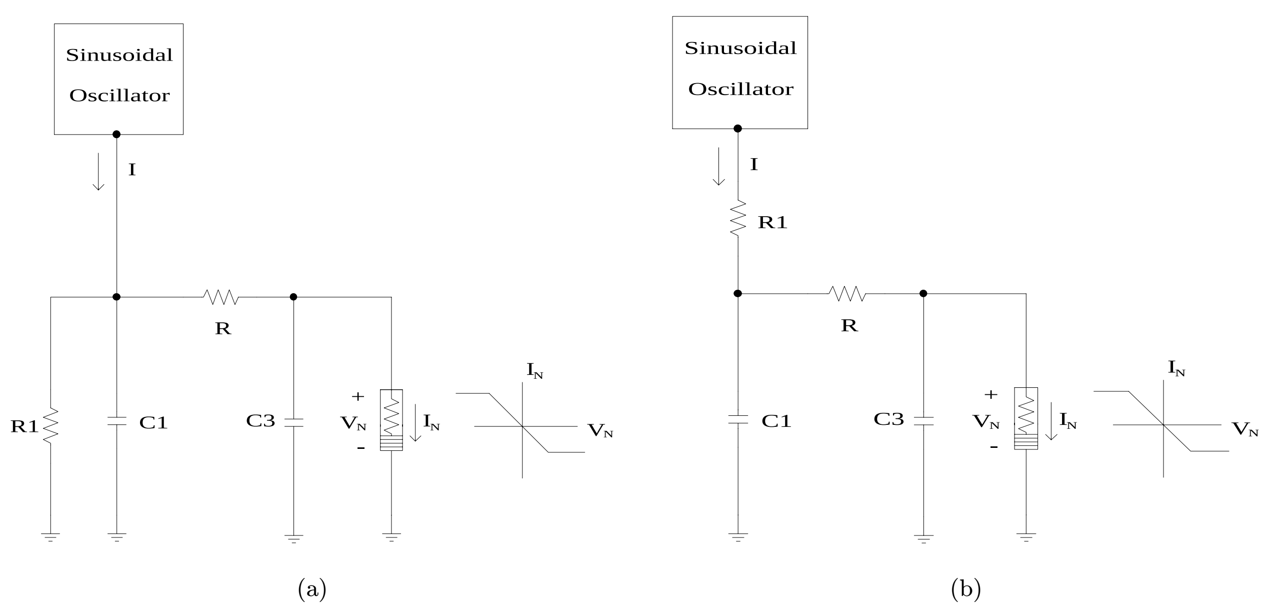

Fig. 2. Chua's circuit based on the oscillator (a) in Fig. 1(a), (b) in Fig. 1(b).

oscillator structure of Fig. 1(b). The result is the circuit shown in Fig. 2(b) which is described by:

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \varepsilon\dot{Z} \end{bmatrix} = \begin{bmatrix} K_1 - K \pm \varepsilon_c & -1 & K \\ nK_2^2 + K_1^2 & -K_1 & 0 \\ K & 0 & -K-a \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -b \end{bmatrix} \quad (8)$$

where a and b are as given by (7b).

The two sets of Eqs. (7) and (8) both reduce to a single system of equations if the condition $K_1 = K_2$ is satisfied in (7) while the condition $K_1 = 0$ is satisfied in (8). For this system, we have performed numerical simulations using a fourth-order Runge–Kutta algorithm after setting, $K = 0.9$, $K_2 = n = \varepsilon = 1$, $A = 4$ and $\varepsilon_c = 0.5$. The resulting double-scroll attractor is plotted in Fig. 3.

For the linearized system, the following sets of eigenvalues were calculated: $(-1.378,\ 0.038 \pm j\,0.808)$ when $a = 0$ and $(3.768,\ -0.534 \pm j\,1.639)$ when $a = -A$. Note that in (7) and (8), the set of parameters $(K_1, K_2, n, \varepsilon_c)$ describes only the sinusoidal oscillator engine while the nonlinearity is represented by the constant A. The link between the engine and the nonlinearity is represented by the two parameters K and ε.

It is thus clear that the mechanism by which oscillations are produced in Chua's circuit is via a generic sinusoidal oscillator. The characteristics of the nonlinear resistor (sketched in Fig. 2) are responsible for switching these oscillations between two parallel surfaces to form the two scrolls. The smaller the value of the switching capacitor C_3, the faster this switching becomes. Recall that since the sinusoidal oscillator is an active block, it is not in general necessary for the nonlinear resistor to be locally active in order to produce chaos [Kennedy, 1995; Elwakil & Kennedy, 1999b]. However, a locally active nonlinear resistor is necessary to produce double-scroll dynamics. Note that the saturation-type characteristics described by (7b) can be replaced by other forms of nonlinearity [Zhong, 1994]. In Fig. 4, the double-scroll projection in the X–Z plane is shown when the nonlinearity is simply given by $f(Z) = A\,\sin(Z)$. The two parameters A and K were set to 3 and 0.55 respectively in this case.

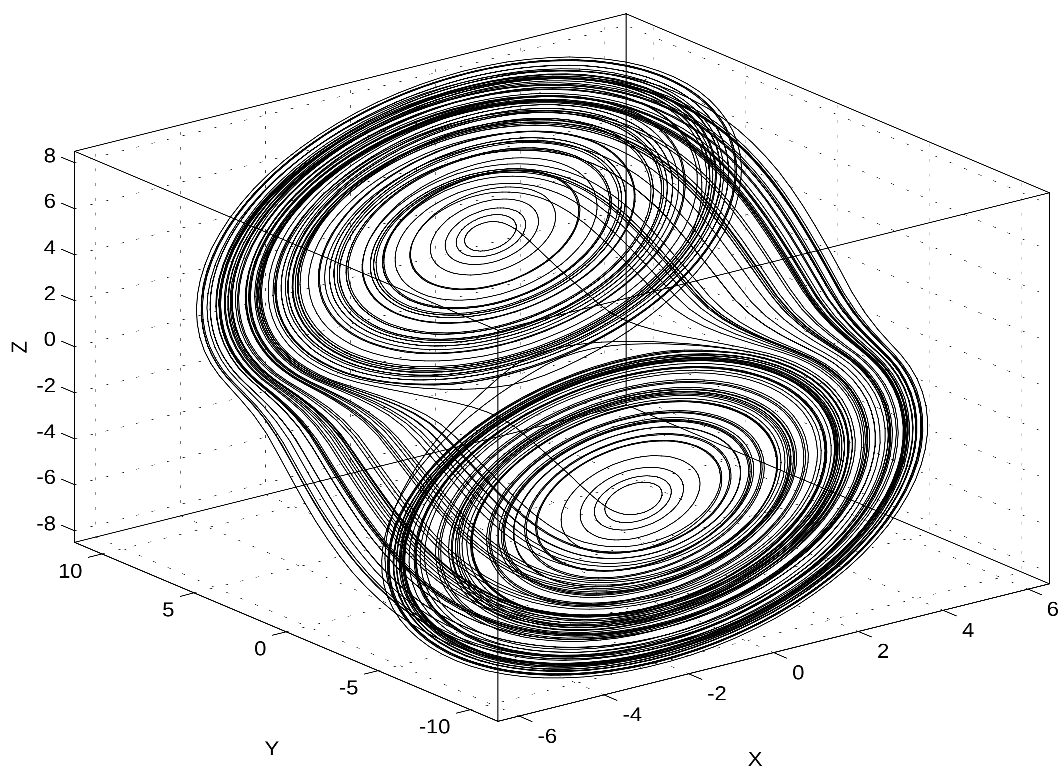

Fig. 3. Double-Scroll Chua's attractor.

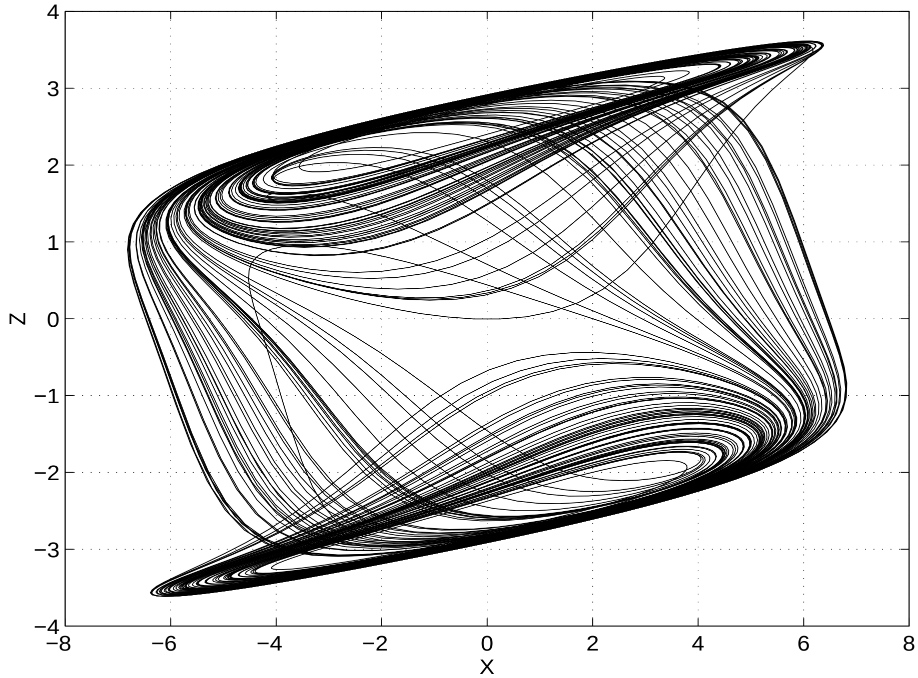

Fig. 4. Projection of the Double-Scroll Chua's attractor when $f(Z) = A\sin(Z)$.

4. Conclusion

We have proposed generic RC realizations of Chua's circuit. With these realizations, we have demonstrated that the core of this chaotic circuit is a sinusoidal oscillator. This confirms our conjecture of [Elwakil & Kennedy, 1999a]. A large number of realizations of this circuit can be found simply by picking any sinusoidal oscillator from literature, or by designing a new one.

Acknowledgments

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References

Cruz, J. M. & Chua, L. O. [1992] "A CMOS IC nonlinear resistor for Chua's circuit," *IEEE Trans. Circuits Syst. I* **39**, 985–995.

Elwakil, A. S. & Kennedy, M. P. [1999a] "A conjecture concerning the design of autonomous chaotic oscillators," *Proc. 7th Int. Specialist Workshop on Nonlinear Dynamics of Electronic Systems NDES'99* Bornholm, pp. 229–231.

Elwakil, A. S. & Kennedy, M. P. [1999b] "Chaotic oscillator configuration using a frequency-dependent negative resistor," *J. Circuits Syst. Comput.* **9** (3&4), 229–242.

Elwakil, A. S. & Kennedy, M. P. [2000] "Improved implementation of Chua's chaotic oscillator using the current feedback op amp," *IEEE Trans. Circuits Syst. I* **47**(1), 76–79.

Kennedy, M. P. [1992] "Robust op amp realization of Chua's circuit," *Frequenz* **46**, 66–80.

Kennedy, M. P. [1995] "On the relationship between the chaotic Colpitts oscillator and Chua's oscillator," *IEEE Trans. Circuits Syst. I* **42**, 376–379.

Matsumoto, T. [1984] "A chaotic attractor from Chua's circuit," *IEEE Trans. Circuits Syst. I* **31**, 1055–1058.

Morgul, O. [1995] "Inductorless realization of Chua's oscillator," *Electron. Lett.* **31**, 1403–1404.

Rodriguez-Vazquez, A. & Delgado-Restituto, M. [1993] "CMOS design of chaotic oscillators using state variables: A monolithic Chua's circuit," *IEEE Trans. Circuits Syst. II* **40**, 596–613.

Zhong, G. [1994] "Implementation of Chua's circuit with a cubic nonlinearity," *IEEE Trans. Circuits Syst. I* **41**, 934–941.