Towards the Realization of Fractional Step Filters

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***Abstract*—In this paper we propose a fractional lowpass transfer function of the order $(n+\alpha)$ where n is an integer and $0 < \alpha < 1$. We show how this filter can be designed using an integer-order transfer function approximation of the fractional-order Laplacian operator s^{α}. A 1^{st} order lowpass filter with fractional steps from 0.1 to 0.9, that is of order 1.1 to 1.9 is given as an example with its characteristics compared to 1^{st} and 2^{nd} order Butterworth filters. PSPICE simulations and experimental results of a prototype filter verify the operation of the fractional step filter.**

I. Introduction

Traditional continuous time filters are electronic circuits used in signal processing for the attenuation of unwanted frequencies or the enhancement of desired ones. These filters are classified as 1^{st}, 2^{nd} or n^{th}order, where n is an integer value. The transfer functions that describe these circuits usually take the form $T(s) = N(s)/D(s)$ where both $N(s)$ and $D(s)$ are polynomials described using the Laplacian operator, s, raised to an integer order; i.e. $s, s^2, \ldots s^n$. While the Laplacian operator in these filters have traditionally only been raised to an integer order, it is mathematically valid to raise to a non-integer order, s^{α}, where $0 < \alpha < 1$; effectively representing a fractional-order system. A fractional derivative may be defined, according to the Riemann-Louville definition[1], [2], [3], as

$$\frac{d^{\alpha}}{dt^{\alpha}} f(t) \equiv D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} d\tau \qquad (1)$$

and $\Gamma(\cdot)$ is the gamma function. Using the Laplace transform, with zero initial conditions, of the derivative above yields

$$L_0 d_t^{\alpha} f(t) = s^{\alpha} F(s) \qquad (2)$$

The concepts of fractional calculus and fractional-order systems have slowly been migrating into the realm of circuit theory and design [4]. The use of fractional calculus in the realm of analog filter design is relatively new [5], [6], [7] with many areas of research still needing to be addressed before the creation of a more general filter design methodology is complete.

In this paper, we propose a new fractional-step low pass filter (FLPF) transfer function, that eliminates the deviation in the passband of previous fractional-step filters [5], [7]. We then utilize a second-order approximation for the fractional Laplacian s^{α} to realize the fractional step filter. We show that the obtained characteristics extend those of the Butterworth filters to the fractional step. An example of a lowpass filter of order $1.1 \to 1.9$ is first simulated using PSPICE and verified experimentally.

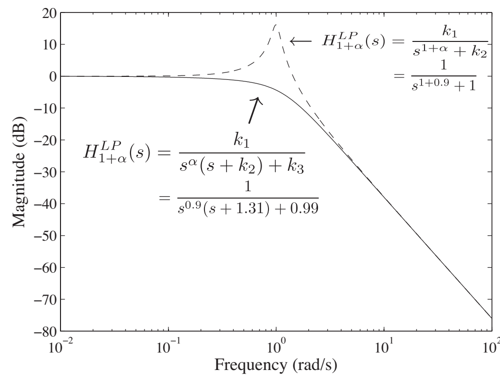

Figure 1. Comparison of FLPF transfer functions of order $(n+\alpha) = 1.9$

II. Proposed Fractional Low Pass Filter

Previous work on fractional-step filters [5] has shown that fractional-step filters are realizable with a reasonable overhead. However, using the proposed transfer functions in [5], [7] yield an undesired peaking in the passband that increases with increasing α. Consider now the FLPF transfer function

$$H_{n+\alpha}^{LP}(s) = \frac{k_1}{s^{\alpha}(s^n + k_2) + k_3} \qquad (3)$$

where n is an integer, $k_{1,2,3}$ are positive constants and $0 < \alpha < 1$. Using this transfer function we can maintain the fractional step through the stopband region while eliminating the undesired passband peaking through appropriate selection of the constants $k_{2,3}$. The elimination of the passband peaking using the proposed FLPF transfer function compared to one previously proposed is shown in Fig. 1 for the case when $n + \alpha = 1.9$ and constants $k_2 = 1.31$ and $k_3 = 0.99$.

By modifying these constants the passband of the magnitude response can be shaped without altering the stopband region. Therefore, through careful selection of $k_{2,3}$ the passband region can be shaped to closer resemble the passband of a Butterworth response while maintaining the desired fractional step through the stopband. The selection of the constants $k_{2,3}$ for minimum passband error can be done numerically; with the cumulative error compared to the Butterworth response through the passband calculated up until the $-3dB$ frequency ω_0; with the values yielding the lowest cumulative error selected for each α. In the section that follows we examine the stability analysis of the proposed equation (3).

A. Stability Analysis

To analyze the stability of the proposed FLPF we transform the transfer function from the s-plane to the W-plane using the

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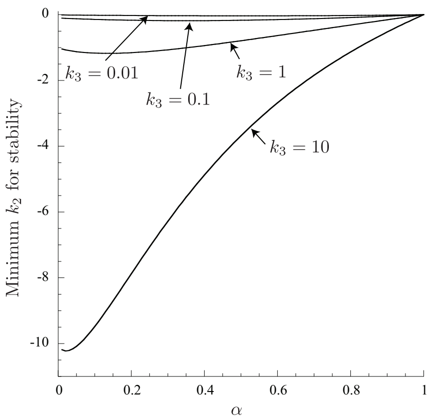


Figure 2. Stability boundaries in the $k_2 - \alpha$ plane for the values of $k_3 = 0.01, 0.1, 1$ and 10

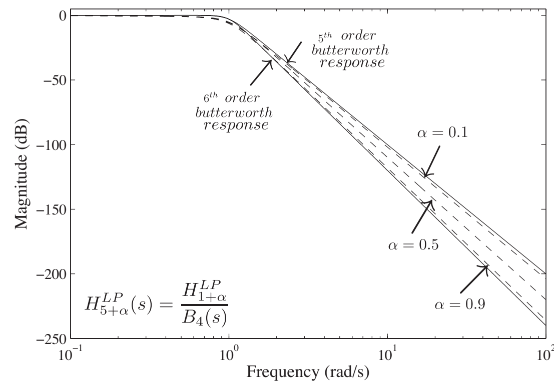


Figure 3. Magnitude response of higher order FLPF with fractional steps from 5.1 to 5.9

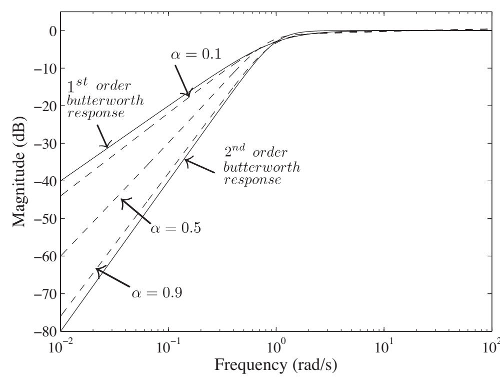


Figure 4. Magnitude response of fractional highpass filters of orders 1.1, 1.5, and 1.9

transformations $s = W^m$ and $\alpha = k/m$ [8]. After applying this transformation, the denominator of (3) becomes:

$$s^{\alpha}(s^n + k_2) + k_3 \rightarrow W^{n \cdot m+k} + k_2 W^k + k_3 \tag{4}$$

which is the characteristic equation in the W-plane that must be solved to ensure that it meets the stability criteria. This stability criteria requires that all $\mid \theta_W \mid > \frac{\pi}{2m}$ for all poles of the characteristic equation, where θ_W is the pole angle in radians. If any of the pole angles do not meet this criteria then the system is unstable. Through numerical analysis of this characteristic equation it was found that the system is always unstable when $n + (k/m) > 2$ regardless of the values of $k_{2,3}$. While for the region $n + (k/m) < 2$ there are always values of $k_{2,3}$ that will ensure stability of the system. An example of the stability boundary in the $k_2 - \alpha$ plane to four values of k_3 are shown in Fig. 2 for the case of $m = 100$ and $n = 1$. This then limits the implementation of filters with a fractional step, without passband peaking, only up to the second order; severely limiting the possibility of higher order fractional step filters.

B. Implementation of Higher Order Fractional Filters

While the highest order that $H_{n+\alpha}^{LP}(s)$ can implement while maintaining stability without the undesired passband peaking is $n+\alpha < 2$, one method of implementing higher order filters with fractional step is to employ $H_{1+\alpha}^{LP}(s)$ which always has a stable region based on $k_{2,3}$ when $0 < \alpha < 1$, divided by a higher-order normalized Butterworth polynomial [9]. This creates a stable higher order fractional step filter of order $n+\alpha$ which can be written as,

$$H_{n+\alpha}^{LP}(s) = \frac{k_1}{s^{\alpha}(s^n + k_2) + k_3} \cong \frac{H_{1+\alpha}^{LP}(s)}{B_{n-1}(s)}; n \geq 2 \tag{5}$$

where $B_n(s)$ is a standard Butterworth polynomial of order n. Using this method we can create higher order filters with a fractional step through the stopband while maintaining the flat passband response. Figure 3 displays the magnitude response of the fractional-step lowpass filter of order $(5+\alpha)$ for values of α =0.1, 0.5, 0.9. Note that the values of $k_{2,3}$ used in the higher order filter to minimize the passband error are the same as those calculated for filters of order $(1 + \alpha)$.

C. Fractional High Pass Filters

To obtain a highpass filter from the its lowpass filter counterpart is straight-forward and implies use of the LP-to-HP transformation. That is, replacing s with $1/s$ in the LP transfer function. This transformation results in a high pass filter with fractional-step through the stopband region. Figure 4 shows a simulation of the magnitude response of this fractional high-pass filter for orders 1.1, 1.5, and 1.9 compared to the 1st and 2nd order Butterworth responses. Note that the flat passband region is also maintained using this lowpass to highpass transformation.

III. Realization of Fractional Low Pass Filter

Since the fractional Laplacian operator s^{α} cannot yet be physically realized, we must make use of an integer order approximation of it so that the FLPF transfer function can be physically realized. Using the continued fraction expansion (CFE) method of [6] we obtain the following approximation for the general Laplacian operator to 2^{nd} order as

$$s^{\alpha} \approx \frac{(\alpha^2 + 3\alpha + 2)s^2 + (8 - 2\alpha^2)s + (\alpha^2 - 3\alpha + 2)}{(\alpha^2 - 3\alpha + 2)s^2 + (8 - 2\alpha^2)s + (\alpha^2 + 3\alpha + 2)} \tag{6}$$

Note that when $\alpha = 0.5$, (6) reverts back to the approximation of $s^{0.5}$ in [6]. Figure 5 shows an example of the magnitude

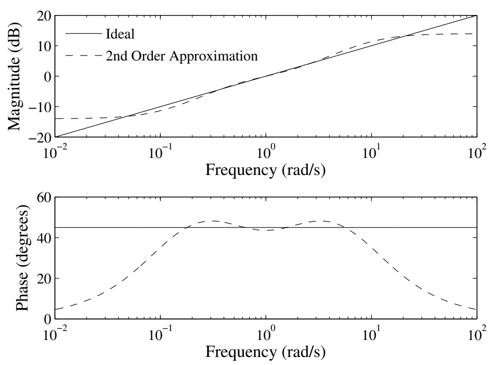


Figure 5. Magnitude and phase error in $s^{0.5}$ using 2^{nd} order s^{α} approximation

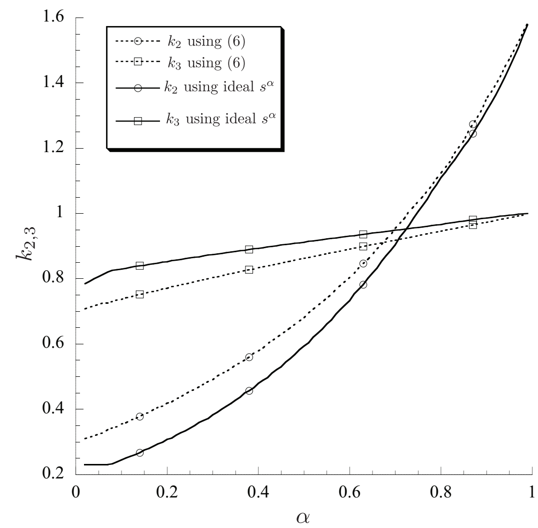


Figure 6. Plot of k_2 and k_3 versus α that yield minimum passband error for the ideal s^{α} and the approximation of s^{α} used in (6).

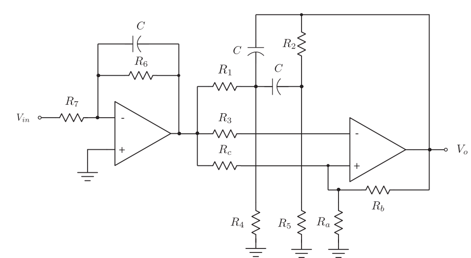


Figure 7. Circuit topology used in the approximation to the fractional low pass filter of order $(1+\alpha)$

and phase of the approximation of the fractional Laplacian operator for the case when $\alpha = 0.5$ compared to the ideal case. For this case it can be observed that for $\omega \in [0.032, 31.53]$ the magnitude error does not exceed $1.375dB$ while for $\omega \in [0.142, 7.00]$ the phase error does not exceed $3.2°$. A 2^{nd} order approximation was selected over a higher order approximation [10], [11], which would have provided a better approximation over a larger frequency range, because it results in lower circuit overhead. Using a second order approximation for the Laplacian operator results in an $(n+2)$ integer order filter to approximate the $(n+\alpha)$ fractional step filter; is much less expensive to implement in hardware over approximations of higher order.

Using this approximation, it can be shown that the transfer function for the FLPF between 1^{st} and 2^{nd} order changes to,

$$\begin{aligned} H^{LP}_{1+\alpha} &= \frac{k_1}{s^{\alpha}(s+k_2)+k_3} \\ &\approxeq \frac{k_1}{a_0}\frac{(a_2s^2+a_1s+a_0)}{s^3+c_0s^2+c_1s+c_2} \end{aligned} \quad (7)$$

where $a_0 = \alpha^2+3\alpha+2$, $a_1 = 8-2\alpha^2$, $a_2 = \alpha^2-3\alpha+2$, $c_0 = (a_1+a_0k_2+a_2k_3)/a_0$, $c_1 = (a_1(k_2+k_3)+a_2)/a_0$ and $c_2 = (a_0k_3+a_2k_2)/a_0$. When using the approximation of the Laplacian operator, the constants $k_{2,3}$ that yield the passband closest to the Butterworth response vary from those used for the ideal transfer function. The values calculated for minimum passband error in the magnitude response for both the ideal and approximate cases of s^{α} are shown in Fig. 6.

A. Circuit Implementation

To realize the transfer function of (7) we can decompose it into first and second order transfer functions that can be realized using first order and single amplifier biquad (SAB) circuit topologies, respectively. For our realization, the two transfer functions we generated were:

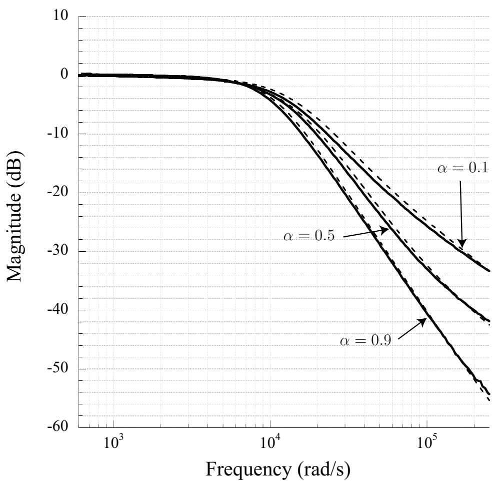
$$\begin{aligned} H(s) &= H_1(s)\cdot H_2(s) \\ &= \frac{1}{s+d_0}\frac{e_0s^2+e_1s+e_2}{s^2+d_1s+d_2} \end{aligned} \quad (8)$$

with the coefficients $d_{0,1,2}$ and $e_{0,1,2}$ determined through the solution of the system of equations made from equating the s terms of (7) and (8). $H_1(s)$, the first order transfer function, is realized using a simple parallel RC network in the feedback portion of an inverting op-amp configuration and $H_2(s)$, the second order transfer function, is realized using the STAR-SAB topology of [12]. The final circuit topology of the fractional low pass filter of order $(1+\alpha)$ is shown in Fig. 7.

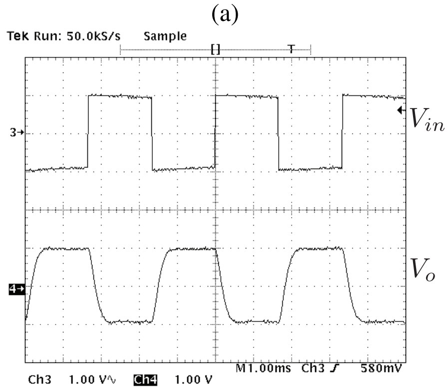
IV. Simulation and Experimental results

The circuit of Fig. 7 was simulated using PSPICE for filters of order $(1+\alpha)$ for the cases α = 0.1, 0.5, and 0.9. These simulations were conducted with general purpose MC1458 op amps ($1MHz$ gain bandwidth product). All time constants were scaled to $0.1ms$ using unit resistors of $1k\Omega$ and capacitors of $0.1uF$. This results in all pole frequencies being shifted from the normalized value of $1rad/s$ to $10^4rad/s$. The theoretical resistor values were rounded to the nearest standard E96 (1%) value. The flat passband response of this filter for the case of $\alpha = 0.1, 0.5$ and 0.9 is shown in Figure 8(a) as dashed lines and the slope can be observed to change with α.

To further verify the simulations of the FLPF, the circuit of Figure 7 was implemented. The components used to implement filters of order 1.1, 1.5, and 1.9 consisted of 1% tolerance resistors and 20% tolerance capacitors. The magnitude



(a)



(b)

Figure 8. (a) Measured and PSPICE simulation results of the magnitude response of the approximated $(1+\alpha)$ order FLPF, shown as solid and dashed lines, respectively. (b) Step response of approximated 1.9 order FLPF.

response of these circuits as measured by a HP4395A Network analyzer are also shown in Figure 8(a), but as solid lines. The two show close agreement confirming that the experimental results match the simulated results. The step response of the 1.9 order filter was also investigated to examine its stability, and is shown in Figure 8(b). We can clearly see from the magnitude response of both the measured and simulated responses that this filter does realize a fractional step through the stopband of the filter without the undesirable peaking of previous fractional step filters. This verifies both the simulation and experimental results proving that the integer order filter can accurately approximate the incremental stepping of a FLPF. Also, as in the simulation results, the measured stopband was not as linear as the magnitude response of the simulated transfer function; which is a result of the deviation from the ideal Laplacian operator associated with using the 2^{nd} order approximation of (6). Comparing the results, we find that the passband attenuation of the experimental and simulated filters are very close to their theoretical values of $-20(1+\alpha)dB/decade$. For comparison, these attenuations are listed in Table I.

Order $(1+\alpha)$	Theoretical (dB/dec)	Simulated (dB/dec)	Experimental (dB/dec)
1.1	-22	-22.48	-22.93
1.5	-30	-29.20	-29.74
1.9	-38	-36.50	-36.44

Table I
THEORETICAL, SIMULATED, AND EXPERIMENTAL STOPBAND ATTENUATIONS OF APPROXIMATED $(1+\alpha)$ ORDER FLPF

V. CONCLUSION

We have proposed a new fractional low-pass filter that demonstrates a fractional step through the stopband while maintaining a flat passband in the magnitude response, thus solving the passband peaking problem of a previously proposed FLPF [5]. Using an integer order approximation of s^α we have simulated and built an integer order filter that demonstrates the fractional step through the stopband. This work while preliminary clearly shows the precise attenuation control available using a fractional filter all the while demonstrating an application of the use of fractional calculus into the design of electronic circuits for signal processing.

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