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A NON-CONSERVATIVE MODEL OF SECOND-ORDER RC SINUSOIDAL OSCILLATORS

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In this work we derive a general non-conservative model for any RC sinusoidal oscillator independent of its particular passive topology or the employed active devices. We consider an arbitrary unknown second-order passive RC network terminated at one port by a negative resistor and proceed to impose oscillation start-up and frequency constraints on a derived state-matrix.

Keywords: Circuit theory; sinusoidal oscillators; RC networks.

1. Introduction

Sinusoidal oscillators are key elements in many electronic systems. Of particular interest here are inductorless RC oscillators, a wide variety of such oscillators have been presented over the years.[1–10] Each oscillator either has a different topology for its passive RC structure (e.g., bridged-T, Twin-T, phase-shift, etc.) or employs a different active device (e.g., op amp, current conveyor, operational transconductance amplifier, etc.) with the aim of obtaining an attractive design feature such as higher oscillation frequency, minimum number of components, all grounded capacitors, etc. However, it was shown in Ref. 11 that no matter what the active device actually is or how it is configured, the source of energy in an oscillator can always be localized as a linear or nonlinear negative resistor. In Ref. 12 we were able to follow-up on that and derive a general yet conservative model for any RC sinusoidal oscillator. By conservative estimates, we assumed that at steady state the passive second-order RC network stores but does not dissipate any part of the energy provided by the negative resistor. This is equivalent to implying that a resistor-less second-order network with only two capacitors, terminated by a negative resistor can sustain sinusoidal oscillations (as confirmed via numerical simulations in Ref. 12) although it is not yet known how such a circuit may physically be realized.

In this letter, we consider the non-conservative case where the RC network is allowed to dissipate part of the energy generated by the negative resistor. Since it is widely known that a canonical minimum component sinusoidal oscillator is one which employs two capacitors and two resistors, we will focus on demonstrating that with only one resistor the network can still oscillate, although it is also not yet known how such a network may be physically realized. A nonlinear state-space model is derived and numerically verified.

2. Proposed Model

Consider the network shown in Fig. 1 composed of an arbitrary second-order passive RC network containing two capacitors (C_1, C_2) and any number of passive resistors. The network is terminated by a linear negative resistor r. The voltage across this negative resistor is generally a function of the two state variables V_{C1} and V_{C2} and can be expressed as $V_r = q_1 V_{C1} + q_2 V_{C2}$, where $q_{1,2}$ are arbitrary positive or negative real numbers. The voltage across any passive resistor R_i inside the network is also generally a function of the two state variables and may be expressed as $V_{R_i} = q_{i_1} V_{C1} + q_{i_2} V_{C2}$ where q_{i_1,i_2} are positive or negative real numbers. Since the only source of energy is the linear negative resistor, current conservation implies

$$C_1 \dot{V}_{C1} + C_2 \dot{V}_{C2} + \sum_{i=1}^{n} \frac{V_{R_i}}{R_i} = I_r = \frac{-V_r}{r}\,, \tag{1}$$

where $C_1 \dot{V}_{C1} = I_{C1}$, $C_2 \dot{V}_{C2} = I_{C2}$ and n is the total number of passive resistors. Defining $X = V_{C1}/V_{\text{ref}}$, $Y = V_{C2}/V_{\text{ref}}$, $\Delta_{ri} = r/R_i$, where V_{ref} is an arbitrary voltage

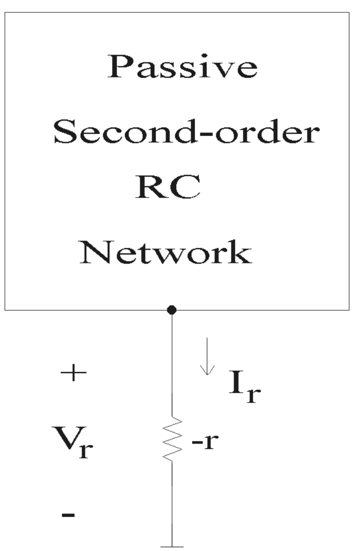

Fig. 1. Structure of a second-order RC oscillator.

reference, and assuming $C_1 = mC_2 = mC$, the above equation becomes

$$m\dot{X} + \dot{Y} + \sum_{i=1}^{n} \Delta_{ri}(q_{i_1}X + q_{i_2}Y) = -q_1 X - q_2 Y\,, \tag{2}$$

where time has also been normalized with respect to rC ($t_n = t/rC$).

On the other hand, the state matrix describing this second-order linear system may be given by

$$\begin{pmatrix} m\dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = (A) \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{3}$$

where X and Y (V_{C1} and V_{C2}) are the two state variables. To admit sinusoidal oscillations implies $tr(A) = 0 \rightarrow a_{22} = -a_{11}/m$. The normalized oscillation frequency[a] is then $\omega_n = \sqrt{|A|} = \sqrt{-(a_{11}/m)^2 - (a_{12}a_{21}/m)}$ with the constraint that either a_{12} or a_{21} must be negative and that $|a_{12}a_{21}| > a_{11}^2/m$. The state matrix above may thus be re-written in terms of the two unknowns a_{11} and a_{12} as

$$\begin{pmatrix} m\dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ -\left(\dfrac{m\omega_n^2 + a_{11}^2/m}{a_{12}}\right) & -a_{11}/m \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \tag{4}$$

from which it is seen that

$$m\dot{X} + \dot{Y} = \left[a_{11} - \frac{m\omega_n^2 + a_{11}^2/m}{a_{12}}\right] X + [a_{12} - a_{11}/m]Y\,. \tag{5}$$

Comparing Eq. (5) with Eq. (2) and solving for a_{11} and a_{12} yields

$$a_{11} = -\frac{m^2\omega_n^2 + m(q_2 + \sum_{i=1}^{n} \Delta_{ri} q_{i_2})(q_1 + \sum_{i=1}^{n} \Delta_{ri} q_{i_1})}{m(q_2 + \sum_{i=1}^{n} \Delta_{ri} q_{i_2}) - q_1 - \sum_{i=1}^{n} \Delta_{ri} q_{i_1}}\,, \tag{6}$$

$$a_{12} = \frac{a_{11}}{m} - q_2 - \sum_{i=1}^{n} \Delta_{ri} q_{i_2}\,, \tag{7}$$

with the constraint $|a_{11}| < m\omega_n/\sqrt{2}$; i.e., $|a_{11}|$ is constrained by the capacitive ratio m and the required normalized oscillation frequency ω_n.

We may simplify the above model by setting the normalized oscillation frequency $\omega_n = 1$. Further, if we consider the common special case of equal valued capacitors ($C_1 = C_2 = C$) (which yields $m = 1$) and the fact that the number of passive resistors R_i is not more than two for a canonic oscillator, we obtain the

[a] Note that the un-normalized oscillation frequency would be $\omega_o = \omega_n/rC$.

following model

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \frac{1}{(q_1 - q_2) + \sum_{i=1}^{2} \Delta_{ri}(q_{i_1} - q_{i2})}$$
$$\times \begin{pmatrix} 1 + \left(q_2 + \sum_{i=1}^{2} \Delta_{ri} q_{i_2}\right)\left(q_1 + \sum_{i=1}^{2} \Delta_{ri} q_{i_1}\right) + \epsilon & 1 + \left(q_2 + \sum_{i=1}^{2} \Delta_{ri} q_{i_2}\right)^2 \\ -1 - \left(q_1 + \sum_{i=1}^{2} \Delta_{ri} q_{i_1}\right)^2 & -1 - \left(q_2 + \sum_{i=1}^{2} \Delta_{ri} q_{i_2}\right)\left(q_1 + \sum_{i=1}^{2} \Delta_{ri} q_{i_1}\right) \end{pmatrix}$$
$$\times \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{8}$$

where ϵ is an error compensation parameter which may be used to locate the pure imaginary eigen pair in the right-half plane ($\epsilon > 0$) and hence guarantee oscillations startup.

3. Nonlinearity and Numerical Simulations

It is well-known that a purely linear system cannot sustain oscillations.[13] The existence of a nonlinearity which guarantees that the start-up unstable operating point at the origin is later shifted to a stable but virtual operating point is essential for bounding the oscillations.[14] As such, most practical realizations of a negative resistor yield monotone $I - V$ characteristics which may be modeled by a cubic nonlinearity of the form[13]

$$I_r = f(V_r) = -aV_r + bV_r^3\,, \tag{9}$$

where a, b are positive constants. Considering Fig. 1 this may be written as

$$\begin{aligned} I_r &= (q_1 V_{C1} + q_2 V_{C2})[b(q_1 V_{C1} + q_2 V_{C2})^2 - a] \\ &= (q_1 V_{C1} + q_2 V_{C2})[b_n - a] = q_{1n} V_{C1} + q_{2n} V_{C2}\,. \end{aligned} \tag{10}$$

We can therefore account for the nonlinearity in the above model (8) by replacing $q_{1,2}$, respectively, with $q_{1n,2n}$ where $q_{1n,2n} = q_{1,2}(-a + b(q_{1,2}X(t) + q_{2,1}Y(t)))$.

In our paper,[12] we have confirmed that the conservative case, where the system does not contain any passive resistance R_i and only contains two capacitors in addition to the negative resistor, can oscillate. Although this situation seems to be impossible to realize circuit-wise as it is not yet known how to connect the two capacitors via non-dissipative zero-order elements to the negative resistor to satisfy the oscillation condition, the mathematical foundation has been laid. Here, we consider for our numerical simulations the case where only **one resistance** R_1 exists; which is also unknown of circuit-wise. As yet, a canonical physically implementable oscillator must contain two passive resistors. However, the results of our numerical simulations below indicate that this is not a mathematical necessity. For simplicity, we set $\Delta_{ri} = \Delta_{r1} = 1$ in Eq. (8).

The set of differential equations to be numerically simulated[b] are then

$$\dot{X} = \left(\frac{1 + \bar{q}_1\bar{q}_2}{\bar{q}_1 - \bar{q}_2} + \epsilon\right)X + \left(\frac{1 + \bar{q}_2^{\,2}}{\bar{q}_1 - \bar{q}_2}\right)Y\,, \tag{11a}$$

$$\dot{Y} = -\left(\frac{1 + \bar{q}_1^{\,2}}{\bar{q}_1 - \bar{q}_2}\right)X - \left(\frac{1 + \bar{q}_1\bar{q}_2}{\bar{q}_1 - \bar{q}_2}\right)Y\,, \tag{11b}$$

where $\bar{q}_1 = -aq_1 + q_{11} + bq_1(q_1X + q_2Y)$ and $\bar{q}_2 = -aq_2 + q_{12} + bq_2(q_2X + q_1Y)$. The behavior of the oscillator is thus dependent on (q_1, q_2) which set the voltage V_r across the negative resistor in proportion to the state variables and (q_{11}, q_{12}) which set the voltage V_R across the positive resistor in proportion to the state variables. At start-up $(X(t) = Y(t) = 0)$ the condition $a \neq (q_{11} - q_{12})/(q_1 - q_2)$ must hold. For simplicity and without loss of generality we further choose $q_{11} = q_{12} = q$ implying $a \neq 0$ which is satisfied for any negative resistor. Figure 2 shows the obtained results with $a = b = q = q_2 = 1$, $q_1 = -1$ and $\epsilon = 0.01$. The upper trace shows $Y(t)$ as oscillations build-up while the lower trace shows $X(t)$ and $Y(t)$ at steady-state. We confirm by setting $b = 0$ that unbounded oscillations are observed indicating that the amplitude limiting function of the nonlinearity, embedded within the simulated model, is functional in Fig. 2. Figure 3 is another simulation showing the limit cycles obtained

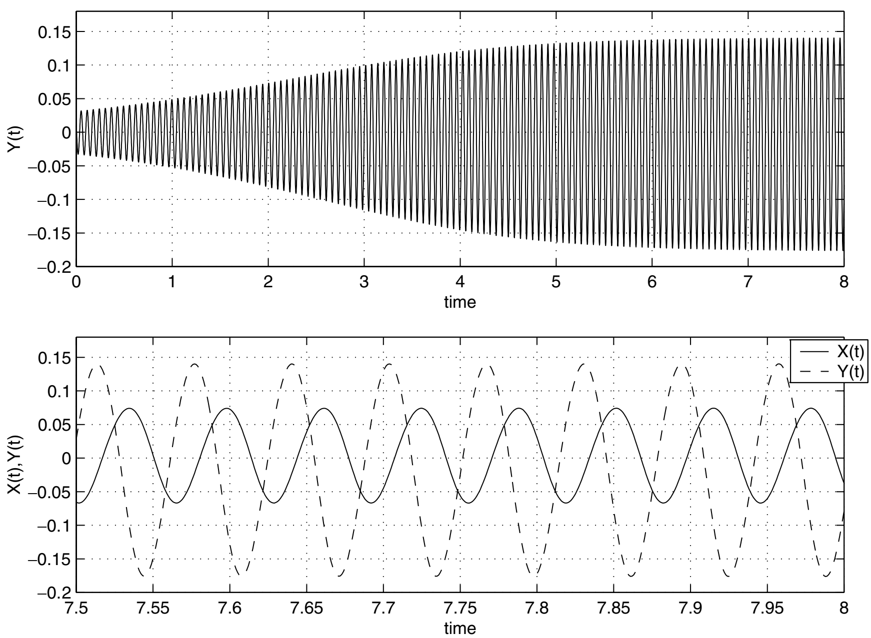

Fig. 2. Time waveforms obtained from numerical simulation.

[b]Using a 4th-order Runge-Kutta algorithm with 0.02 step size.

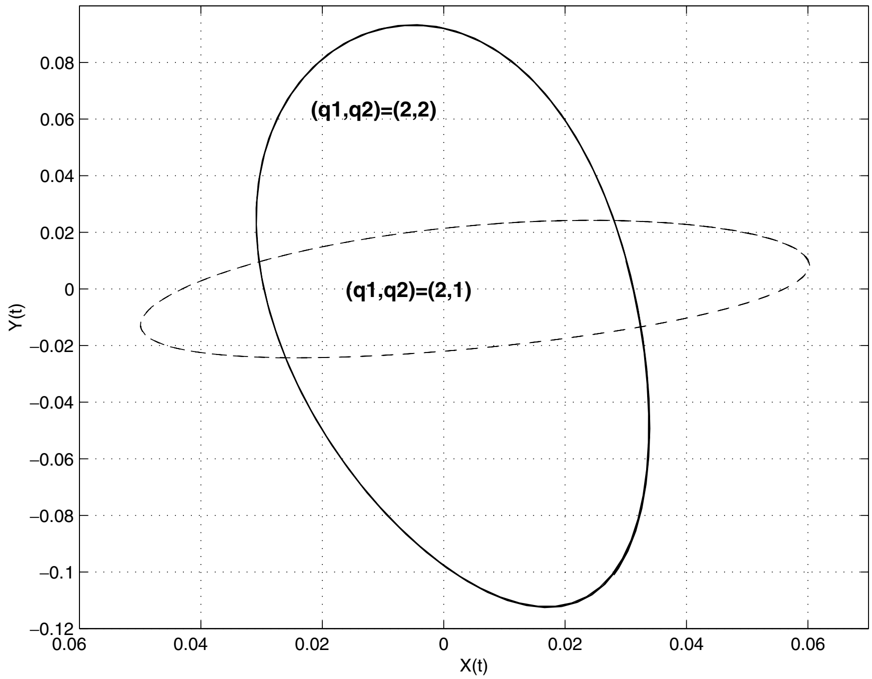

Fig. 3. Limit cycles showing the $X - Y$ phase change with changing q_2.

respectively with $(q_1, q_2) = (-2, 2)$ and $(-2, 1)$; all other parameters being the same as for Fig. 2. The figure shows that the phase lead/lag relationship between $X(t)$ and $Y(t)$ can be changed.

4. Conclusion

A non-conservative model for second-order RC sinusoidal oscillators was proposed and numerically verified in the selective case where the passive network contains two capacitors and only one positive resistor. Although a circuit design of such a structure has never been reported, it is mathematically valid.

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